

Penalty Robin–Robin domain decomposition methods for unilateral multibody contact problems of elasticity: Convergence results

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Abstract

The paper is devoted to penalty Robin–Robin domain decomposition methods (DDMs), proposed by us for the solution of unilateral multibody contact problems of elasticity. These DDMs are based on the penalty method for variational inequalities and some stationary and nonstationary iterative methods for nonlinear variational equations. The main result of the paper is that we give mathematical justification of proposed DDMs and prove theorems about their convergence. We also provide numerical investigations of the efficiency of these methods using finite element approximations.

Key words: elasticity, multibody contact, variational inequalities, penalty method, iterative methods, domain decomposition

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1 Introduction

Contact problems of elasticity are widely used in many fields of science and engineering, especially in machine science, structural mechanics, geology and biomechanics. The brief overview of existing numerical and analytical methods for the solution of contact problems can be found in [1, 2].

Efficient approach to the solution of multibody contact problems is the use of domain decomposition methods (DDMs).

DDMs are well developed for the solution of linear boundary value problems, particularly for linear Poisson and linear elasticity problems [3, 4, 5].

The construction of DDMs for unilateral contact problems, which are nonlinear, are much more complicated. Among domain decomposition methods for unilateral two-body contact problems obtained on continuous level, one should

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mention Signorini–Neumann [6, 7], Signorini–Dirichlet [8, 9] and Signorini–Signorini [10] iterative algorithms. All of these methods in each iteration require to solve a nonlinear one-sided contact problem with a rigid surface (Signorini problem) for one of the bodies, and a linear elasticity problem with Neumann [6, 7] or Dirichlet [8, 9] boundary conditions on possible contact area for other body, or require to solve nonlinear Signorini problems for both of the bodies [10]. Moreover, to increase the convergence rate of Signorini–Dirichlet and Signorini–Signorini algorithms, it is recommended to perform an additional iteration, in which we have to solve linear elasticity problems with Neumann boundary conditions for both of the bodies [8, 10].

Domain decomposition method presented in work [11] for two-body unilateral contact problem, is also obtained on continuous level. It is based on the use of augmented Lagrangian variational formulation and Uzawa block relaxation method. This domain decomposition method in each iteration require to solve linear elasticity problems for both of the bodies.

On contrary, DDMs can be constructed on discrete level, after a discretization of corresponding continuous boundary-value problem. Among discrete DDMs for unilateral contact problems, one should mark out substructuring and FETI methods [12, 13, 14, 15].

In works [16, 17, 18, 19] we have proposed on continuous level a class of penalty parallel Robin–Robin type domain decomposition methods for the solution of unilateral multibody contact problems of elasticity. These methods are based on penalty method for variational inequalities and some stationary and nonstationary iterative methods for nonlinear variational equations. In each iteration of proposed DDMs we have to solve in parallel some linear variational equations in subdomains, which correspond to elasticity problems with Robin boundary conditions, prescribed on some subareas of possible contact zones. These DDMs do not require the solution of nonlinear one-sided contact problems in each step.

The main result of this paper is that we have proved theorems about convergence of proposed penalty Robin–Robin domain decomposition methods. The paper is organized as follows. In section 2 the classical formulation of multibody contact problem in the form of the system of second order elliptic partial differential equations with inequality and equality constraints is given. In section 3 we consider the variational formulation of this problem in the form of convex minimization problem and in the form of elliptic variational inequality at closed convex set, and formulate theorem about a unique existence of the solution of this inequality. In section 4 we use the penalty method to reduce the variational inequality to an unconstrained minimization problem, which is equivalent to a nonlinear variational equation in the whole space. Later, we prove a theorem about a unique existence of the solution of the penalty variational equation and a theorem about strong convergence of this solution to the solution of initial variational inequality. In section 5 we consider stationary and nonstationary iterative methods for the solution of abstract nonlinear variational equations in reflexive Banach spaces. We prove theorems about convergence of these methods, and show that the convergence rate of stationary methods in some energy norm is linear. We also prove theorem about stability of stationary iterative methods to errors which may occur in each iteration. In section 6 we present parallel stationary and nonstationary penalty Robin–Robin domain decomposition methods for the solution of nonlinear penalty variational

equations of multibody unilateral contact problems. We prove theorem about convergence of these methods, and show that the convergence rate of stationary Robin–Robin methods in some energy norm is linear. In section 7 we provide numerical investigations of proposed domain decomposition methods using finite element approximations. The penalty parameter and mesh refinement influence on the numerical solution, as well as the dependence of the convergence rate of domain decomposition methods on iterative parameters are investigated. In conclusion section we summarize all results presented in the paper.

2 Formulation of unilateral multibody contact problem

Introduce the Cartesian coordinate system $Ox_1x_2x_3$ with basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and consider the problem of frictionless unilateral contact of N elastic bodies $\Omega_\alpha \subset \mathbb{R}^3$ with piecewise smooth boundaries $\Gamma_\alpha = \partial\Omega_\alpha$, $\alpha = 1, 2, \dots, N$ (fig. 1). Denote $\Omega = \bigcup_{\alpha=1}^N \Omega_\alpha$.

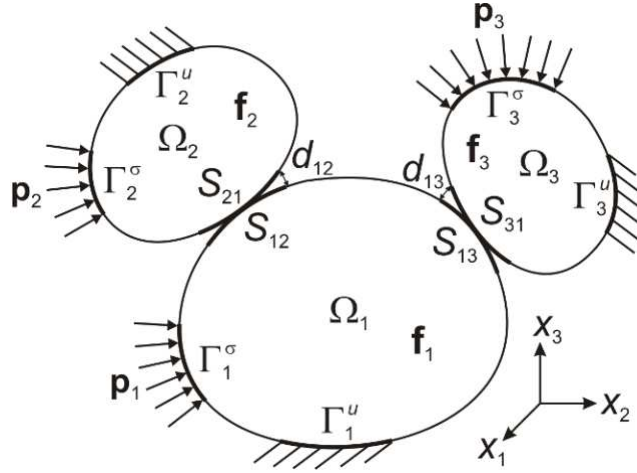


Fig 1. Unilateral contact of several bodies

The stress-strain state in point $\mathbf{x} = (x_1, x_2, x_3)^T$ of each solid Ω_α is described by the displacement vector $\mathbf{u}_\alpha(\mathbf{x}) = u_{\alpha i}(\mathbf{x}) \mathbf{e}_i$, the symmetric tensor of strains $\hat{\varepsilon}_\alpha = \varepsilon_{\alpha ij} \mathbf{e}_i \mathbf{e}_j$, and the tensor of stresses $\hat{\sigma}_\alpha = \sigma_{\alpha ij} \mathbf{e}_i \mathbf{e}_j$. These quantities satisfy Cauchy relations, Hook's Law and the equilibrium equations:

$$\varepsilon_{\alpha ij}(\mathbf{x}) = \frac{1}{2} \left(\frac{\partial u_{\alpha i}(\mathbf{x})}{\partial x_j} + \frac{\partial u_{\alpha j}(\mathbf{x})}{\partial x_i} \right), \quad \mathbf{x} \in \Omega_\alpha, \quad i, j = 1, 2, 3, \quad (1)$$

$$\sigma_{\alpha ij}(\mathbf{x}) = \sum_{k,l=1}^3 C_{\alpha ijkl}(\mathbf{x}) \varepsilon_{\alpha kl}(\mathbf{x}), \quad \mathbf{x} \in \Omega_\alpha, \quad i, j = 1, 2, 3, \quad (2)$$

$$\sum_{j=1}^3 \frac{\partial \sigma_{\alpha ij}(\mathbf{x})}{\partial x_j} + f_{\alpha i}(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega_\alpha, \quad i = 1, 2, 3, \quad (3)$$

where $f_{\alpha i}(\mathbf{x})$ are the components of the volume forces vector $\mathbf{f}_\alpha(\mathbf{x}) = f_{\alpha i}(\mathbf{x}) \mathbf{e}_i$.

The elastic coefficients $C_{\alpha ijkl}(\mathbf{x})$ are measurable, symmetric, and bounded with constants $0 < b \leq d < \infty$ in the following sense [20]:

$$b \sum_{i,j=1}^3 \varepsilon_{\alpha ij}^2(\mathbf{x}) \leq \sum_{i,j,k,l=1}^3 C_{\alpha ijkl}(\mathbf{x}) \varepsilon_{\alpha ij}(\mathbf{x}) \varepsilon_{\alpha kl}(\mathbf{x}) \leq d \sum_{k,l=1}^3 \varepsilon_{\alpha kl}^2(\mathbf{x}). \quad (4)$$

Suppose that the boundary Γ_α of each solid consists of three parts: Γ_α^u , Γ_α^σ , S_α , such that $\Gamma_\alpha = \Gamma_\alpha^u \cup \Gamma_\alpha^\sigma \cup S_\alpha$, $\Gamma_\alpha^u \cap \Gamma_\alpha^\sigma \cap S_\alpha = \emptyset$, $\Gamma_\alpha^u \neq \emptyset$, $\Gamma_\alpha^u = \overline{\Gamma_\alpha^u}$, $S_\alpha \neq \emptyset$. Boundary $S_\alpha = \bigcup_{\beta \in B_\alpha} S_{\alpha\beta}$ is the possible contact area of body Ω_α with other bodies, $S_{\alpha\beta}$ is the possible contact area of body Ω_α with body Ω_β , and $B_\alpha \subset \{1, 2, \dots, N\}$ is the set of the indices of all bodies which are in contact with body Ω_α .

On each boundary Γ_α let us introduce the local orthonormal coordinate system $\boldsymbol{\xi}_\alpha$, $\boldsymbol{\zeta}_\alpha$, \mathbf{n}_α , where \mathbf{n}_α is outer unit normal to Γ_α . Then the vectors of displacements and stresses on the boundary can be written in the following way:

$$\mathbf{u}_\alpha(\mathbf{x}) = u_{\alpha\xi}(\mathbf{x})\boldsymbol{\xi}_\alpha + u_{\alpha\zeta}(\mathbf{x})\boldsymbol{\zeta}_\alpha + u_{\alpha n}(\mathbf{x})\mathbf{n}_\alpha, \quad \mathbf{x} \in \Gamma_\alpha,$$

$$\boldsymbol{\sigma}_\alpha(\mathbf{x}) = \hat{\boldsymbol{\sigma}}_\alpha(\mathbf{x}) \cdot \mathbf{n}_\alpha = \sigma_{\alpha\xi}(\mathbf{x})\boldsymbol{\xi}_\alpha + \sigma_{\alpha\zeta}(\mathbf{x})\boldsymbol{\zeta}_\alpha + \sigma_{\alpha n}(\mathbf{x})\mathbf{n}_\alpha, \quad \mathbf{x} \in \Gamma_\alpha.$$

We assume that the surfaces $S_{\alpha\beta} \subset \Gamma_\alpha$ and $S_{\beta\alpha} \subset \Gamma_\beta$ are sufficiently close [21]. Therefore $\mathbf{n}_\alpha(\mathbf{x}) \approx -\mathbf{n}_\beta(\mathbf{x}')$, where $\mathbf{x}' = P(\mathbf{x}) \in S_{\beta\alpha}$ is the projection of point $\mathbf{x} \in S_{\alpha\beta}$ on the surface $S_{\beta\alpha}$. We denote by $d_{\alpha\beta}(\mathbf{x}) = \pm \|\mathbf{x} - \mathbf{x}'\|_2$ the distance in \mathbb{R}^3 between bodies Ω_α and Ω_β before the deformation. The sign of $d_{\alpha\beta}(\mathbf{x})$ depends on a statement of the problem.

On the part Γ_α^u the kinematical (Dirichlet) boundary conditions are prescribed:

$$\mathbf{u}_\alpha(\mathbf{x}) = \mathbf{z}_\alpha(\mathbf{x}), \quad \mathbf{x} \in \Gamma_\alpha^u, \quad (5)$$

and on the part Γ_α^σ we consider the static (Neumann) boundary conditions

$$\boldsymbol{\sigma}_\alpha(\mathbf{x}) = \mathbf{p}_\alpha(\mathbf{x}), \quad \mathbf{x} \in \Gamma_\alpha^\sigma, \quad (6)$$

where $\mathbf{z}_\alpha = z_{\alpha\xi}(\mathbf{x})\boldsymbol{\xi}_\alpha + z_{\alpha\zeta}(\mathbf{x})\boldsymbol{\zeta}_\alpha + z_{\alpha n}(\mathbf{x})\mathbf{n}_\alpha$ and $\mathbf{p}_\alpha = p_{\alpha\xi}(\mathbf{x})\boldsymbol{\xi}_\alpha + p_{\alpha\zeta}(\mathbf{x})\boldsymbol{\zeta}_\alpha + p_{\alpha n}(\mathbf{x})\mathbf{n}_\alpha$ are given boundary displacements and stresses.

Further, for the simplicity of variational formulations and proofs, we assume that all bodies are rigidly fixed on the surface Γ_α^u , i.e.

$$\mathbf{z}_\alpha(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma_\alpha^u. \quad (7)$$

On the possible contact areas $S_{\alpha\beta}$, $\alpha = 1, 2, \dots, N$, $\beta \in B_\alpha$ the following unilateral contact conditions hold:

absence of extension

$$\sigma_{\alpha n}(\mathbf{x}) = \sigma_{\beta n}(\mathbf{x}') \leq 0, \quad (8)$$

absence of friction

$$\sigma_{\alpha\xi}(\mathbf{x}) = \sigma_{\beta\xi}(\mathbf{x}') = 0, \quad \sigma_{\alpha\zeta}(\mathbf{x}) = \sigma_{\beta\zeta}(\mathbf{x}') = 0, \quad (9)$$

mutual nonpenetration of the bodies

$$u_{\alpha n}(\mathbf{x}) + u_{\beta n}(\mathbf{x}') \leq d_{\alpha\beta}(\mathbf{x}), \quad (10)$$

contact alternative

$$(u_{\alpha n}(\mathbf{x}) + u_{\beta n}(\mathbf{x}') - d_{\alpha\beta}(\mathbf{x})) \sigma_{\alpha n}(\mathbf{x}) = 0, \quad (11)$$

where $\mathbf{x} \in S_{\alpha\beta}$, $\mathbf{x}' = P(\mathbf{x}) \in S_{\beta\alpha}$.

The system of second order partial differential equations (1) – (3) with boundary conditions (5) – (11) is the mathematical formulation of frictionless unilateral multibody contact problem of elasticity.

Note that the contact problem (1) – (3), (5) – (11) is nonlinear, since the real contact areas are unknown.

3 Variational formulation of the contact problem

Let us give the weak formulation of the contact problem (1) – (3), (5) – (11) in the form of variational equality and convex minimization problem.

For each body Ω_α , $\alpha = 1, 2, \dots, N$ consider Sobolev space $V_\alpha = [H^1(\Omega_\alpha)]^3$ with scalar product $(\mathbf{u}_\alpha, \mathbf{v}_\alpha)_{V_\alpha} = \sum_{i=1}^3 \int_{\Omega_\alpha} (u_{\alpha i} v_{\alpha i} + \sum_{j=1}^3 \frac{\partial u_{\alpha i}}{\partial x_j} \frac{\partial v_{\alpha i}}{\partial x_j}) d\Omega$,

$\mathbf{u}_\alpha, \mathbf{v}_\alpha \in V_\alpha$ and norm $\|\mathbf{u}_\alpha\|_{V_\alpha} = \sqrt{(\mathbf{u}_\alpha, \mathbf{u}_\alpha)_{V_\alpha}}$, $\mathbf{u}_\alpha \in V_\alpha$.

Introduce the following closed subspace in V_α :

$$V_\alpha^0 = \{ \mathbf{u}_\alpha : \mathbf{u}_\alpha \in V_\alpha, \text{Tr}_\alpha^u(\mathbf{u}_\alpha) = 0 \text{ on } \Gamma_\alpha^u \}, \quad (12)$$

where $\text{Tr}_\alpha^u : V_\alpha \rightarrow [H^{1/2}(\Gamma_\alpha^u)]^3$ is surjective, linear and continuous trace operator [20, 22]. Space V_α^0 is a Hilbert space with the same scalar product and norm as in V_α .

Consider the space V_0 , which is the direct product of spaces V_α^0 :

$$V_0 = V_1^0 \times \dots \times V_N^0 = \left\{ \mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_N)^T, \mathbf{u}_\alpha \in V_\alpha^0, \alpha = 1, 2, \dots, N \right\}, \quad (13)$$

and define in it the scalar product and norm: $(\mathbf{u}, \mathbf{v})_{V_0} = \sum_{\alpha=1}^N (\mathbf{u}_\alpha, \mathbf{v}_\alpha)_{V_\alpha}$, $\|\mathbf{u}\|_{V_0} = \sqrt{(\mathbf{u}, \mathbf{u})_{V_0}}$, $\mathbf{u}, \mathbf{v} \in V_0$. Note, that Hilbert space V_0 is closed reflexive Banach space.

Now, let us introduce the closed convex set of all displacement vectors in V_0 which satisfy nonpenetration contact conditions (10):

$$K = \{ \mathbf{u} : \mathbf{u} \in V_0, u_{\alpha n} + u_{\beta n} \leq d_{\alpha\beta} \text{ on } S_{\alpha\beta}, \{ \alpha, \beta \} \in Q \}, \quad (14)$$

where $Q = \{ \{ \alpha, \beta \} : \alpha \in \{1, 2, \dots, N\}, \beta \in B_\alpha \}$ is the set of all possible unordered pairs of the subscripts of bodies which are in contact with each other, and $d_{\alpha\beta} \in H_{00}^{1/2}(\Sigma_\alpha)$, $\{ \alpha, \beta \} \in Q$, $\Sigma_\alpha = \text{int}(\Gamma_\alpha \setminus \Gamma_\alpha^u)$, $\alpha = 1, 2, \dots, N$.

The quantities $u_{\alpha n}$, $\alpha = 1, 2, \dots, N$ in (14) have to be understood in the following way

$$u_{\alpha n} = \mathbf{n}_\alpha \cdot \text{Tr}_\alpha^0(\mathbf{u}_\alpha), \mathbf{u}_\alpha \in V_\alpha^0,$$

where $\text{Tr}_\alpha^0 : V_\alpha^0 \rightarrow [H_{00}^{1/2}(\Sigma_\alpha)]^3$ is surjective, linear and continuous trace operator on surface $\Sigma_\alpha = \text{int}(\Gamma_\alpha \setminus \Gamma_\alpha^u)$ [20, 22], and $\mathbf{n}_\alpha \in [L_2(\Sigma_\alpha)]^3$.

Note, that all equalities and inequalities in spaces L_2 , $H^{1/2}$, $H_{00}^{1/2}$ and H^1 hold almost everywhere.

Since set K is a closed convex subset of Hilbert space V_0 [20, 21, 23], it is weakly closed [24].

In space $V_0 \times V_0$ consider bilinear form $A(\mathbf{u}, \mathbf{v})$, which represents the total deformation energy of the system of bodies:

$$A(\mathbf{u}, \mathbf{v}) = \sum_{\alpha=1}^N a_{\alpha}(\mathbf{u}_{\alpha}, \mathbf{v}_{\alpha}), \quad \mathbf{u}, \mathbf{v} \in V_0, \quad (15)$$

$$a_{\alpha}(\mathbf{u}_{\alpha}, \mathbf{v}_{\alpha}) = \int_{\Omega_{\alpha}} \hat{\boldsymbol{\sigma}}_{\alpha}(\mathbf{u}_{\alpha}) : \hat{\boldsymbol{\varepsilon}}_{\alpha}(\mathbf{v}_{\alpha}) d\Omega, \quad \mathbf{u}_{\alpha}, \mathbf{v}_{\alpha} \in V_{\alpha}^0. \quad (16)$$

In space V_0 define linear form $L(\mathbf{v})$, which is equal to the external forces work:

$$L(\mathbf{v}) = \sum_{\alpha=1}^N l_{\alpha}(\mathbf{v}_{\alpha}), \quad \mathbf{v} \in V_0, \quad (17)$$

$$l_{\alpha}(\mathbf{v}_{\alpha}) = \int_{\Omega_{\alpha}} \mathbf{f}_{\alpha} \cdot \mathbf{v}_{\alpha} d\Omega + \int_{\Gamma_{\alpha}^{\sigma}} \mathbf{p}_{\alpha} \cdot \text{Tr}_{\alpha}^0(\mathbf{v}_{\alpha}) dS, \quad \mathbf{v}_{\alpha} \in V_{\alpha}^0. \quad (18)$$

where $\mathbf{f}_{\alpha} \in [L_2(\Omega_{\alpha})]^3$, $\mathbf{p}_{\alpha} \in [H_{00}^{-1/2}(\Sigma_{\alpha})]^3$, $\alpha = 1, 2, \dots, N$.

Lemma 1. *If $\Gamma_{\alpha} = \partial\Omega_{\alpha}$, $\alpha = 1, 2, \dots, N$ are piecewise smooth, $\Gamma_{\alpha}^u \neq \emptyset$, $\Gamma_{\alpha}^u = \overline{\Gamma_{\alpha}^u}$, $\mathbf{f}_{\alpha} \in [L_2(\Omega_{\alpha})]^3$, $\mathbf{p}_{\alpha} \in [H_{00}^{-1/2}(\Sigma_{\alpha})]^3$, and condition (4) holds, then the bilinear form A is symmetric, continuous and coercive, and the linear form L is continuous, i.e.*

$$(\forall \mathbf{u}, \mathbf{v} \in V_0) \{A(\mathbf{u}, \mathbf{v}) = A(\mathbf{v}, \mathbf{u})\}, \quad (19)$$

$$(\exists M > 0) (\forall \mathbf{u}, \mathbf{v} \in V_0) \{ |A(\mathbf{u}, \mathbf{v})| \leq M \|\mathbf{u}\|_{V_0} \|\mathbf{v}\|_{V_0} \}, \quad (20)$$

$$(\exists B > 0) (\forall \mathbf{u} \in V_0) \{ A(\mathbf{u}, \mathbf{u}) \geq B \|\mathbf{u}\|_{V_0}^2 \}, \quad (21)$$

$$(\exists T > 0) (\forall \mathbf{v} \in V_0) \{ |L(\mathbf{v})| \leq T \|\mathbf{v}\|_{V_0} \}. \quad (22)$$

Proof. The symmetry of bilinear form $A(\mathbf{u}, \mathbf{v})$ follows from the symmetry of elastic constants in the Hook's law (2), and its continuity and coercivity follows from the continuity and coercivity of bilinear forms $a_{\alpha}(\mathbf{u}_{\alpha}, \mathbf{v}_{\alpha})$ for each body. The continuity of $L(\mathbf{v})$ follows from the continuity of linear forms $l_{\alpha}(\mathbf{v}_{\alpha})$, which can be proved using trace theorems [20]. The detailed proof of this Lemma can be found in [19]. \square

According to [21, 23], the original contact problem (1) – (3), (5) – (11) has an alternative weak formulation as the convex minimization problem of the quadratic functional on the set K :

$$F(\mathbf{u}) = \frac{1}{2} A(\mathbf{u}, \mathbf{u}) - L(\mathbf{u}) \rightarrow \min_{\mathbf{u} \in K}. \quad (23)$$

Using the general theory of variational inequalities [22, 24, 25] we have proved in [21, 23] the next theorem.

Theorem 1. *Suppose that bilinear form A and linear form L satisfy properties (19) – (22), and set K is convex closed subset of Hilbert space V_0 . Then*

the minimization problem (23) has a unique solution $\mathbf{u} \in K$, and this problem is equivalent to the following variational inequality:

$$F'(\mathbf{u}, \mathbf{v} - \mathbf{u}) = A(\mathbf{u}, \mathbf{v} - \mathbf{u}) - L(\mathbf{v} - \mathbf{u}) \geq 0, \quad \forall \mathbf{v} \in K. \quad (24)$$

4 Penalty variational formulation of the problem

To obtain a minimization problem in original space V_0 , we apply a penalty method [24, 25] to convex minimization problem (23).

For the violation of nonpenetration conditions we use the penalty in the following form [17, 20]:

$$J_\theta(\mathbf{u}) = \frac{1}{2\theta} \sum_{\{\alpha, \beta\} \in Q} \int_{S_{\alpha\beta}} \left[(d_{\alpha\beta} - u_{\alpha n} - u_{\beta n})^- \right]^2 dS, \quad (25)$$

where $\theta > 0$ is a penalty parameter, $y^- = \min\{0, y\}$.

Let us consider the following minimization problem with penalty in space V_0 :

$$F_\theta(\mathbf{u}) = \frac{1}{2} A(\mathbf{u}, \mathbf{u}) - L(\mathbf{u}) + J_\theta(\mathbf{u}) \rightarrow \min_{\mathbf{u} \in V_0}. \quad (26)$$

Note, that the introduction of penalty corresponds to the introduction of conditional intermediate Winkler layer between bodies with stiffness coefficient $1/\theta$. The quantity $\sigma_{\alpha\beta n} = \sigma_{\alpha n} = \sigma_{\beta n} = (d_{\alpha\beta} - u_{\alpha n} - u_{\beta n})^-/\theta$ has a sense of the normal contact stress between bodies Ω_α and Ω_β , and the penalty $J_\theta(\mathbf{u})$ represents the total normal contact stresses work.

Now consider the properties of penalty term (25) in more detail. Functional $J_\theta(\mathbf{u})$ is nonnegative

$$(\forall \mathbf{u} \in V_0) \{J_\theta(\mathbf{u}) \geq 0\}, \quad (27)$$

and Gateaux differentiable in V_0 :

$$J'_\theta(\mathbf{u}, \mathbf{v}) = -\frac{1}{\theta} \sum_{\{\alpha, \beta\} \in Q} \int_{S_{\alpha\beta}} (d_{\alpha\beta} - u_{\alpha n} - u_{\beta n})^- (v_{\alpha n} + v_{\beta n}) dS, \quad (28)$$

Furthermore, the Gateaux differential $J'_\theta(\mathbf{u}, \mathbf{v})$ is linear in \mathbf{v} .

Lemma 2. *If surfaces $S_{\alpha\beta}$, $\{\alpha, \beta\} \in Q$ are piecewise smooth, and $d_{\alpha\beta} \in H_{00}^{1/2}(\Sigma_\alpha)$, then $J'_\theta(\mathbf{u}, \mathbf{v})$ satisfy the following properties:*

$$(\forall \mathbf{u} \in V_0) \left(\exists \tilde{R} > 0 \right) (\forall \mathbf{v} \in V_0) \left\{ |J'_\theta(\mathbf{u}, \mathbf{v})| \leq \tilde{R} \|\mathbf{v}\|_{V_0} \right\}, \quad (29)$$

$$(\forall \mathbf{u}, \mathbf{v} \in V_0) \{J'_\theta(\mathbf{u} + \mathbf{v}, \mathbf{v}) - J'_\theta(\mathbf{u}, \mathbf{v}) \geq 0\}, \quad (30)$$

$$(\exists D > 0) (\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V_0) \{ |J'_\theta(\mathbf{u} + \mathbf{w}, \mathbf{v}) - J'_\theta(\mathbf{u}, \mathbf{v})| \leq D \|\mathbf{v}\|_{V_0} \|\mathbf{w}\|_{V_0} \}. \quad (31)$$

Proof. It is obvious to see, that functional $J'_\theta(\mathbf{u}, \mathbf{v})$ is linear in \mathbf{v} .

At first, let us show the satisfaction of property (29). Let us write $J'_\theta(\mathbf{u}, \mathbf{v})$ in the extended form: $J'_\theta(\mathbf{u}, \mathbf{v}) = \sum_{\{\alpha, \beta\} \in Q} j'_{\alpha\beta}(\mathbf{u}, \mathbf{v})$, where

$$j'_{\alpha\beta}(\mathbf{u}, \mathbf{v}) = -\frac{1}{\theta} \left(\int_{S_{\alpha\beta}} g_{\alpha\beta}(\mathbf{u}) \mathbf{n}_\alpha \cdot \text{Tr}_\alpha^0(\mathbf{v}_\alpha) dS + \int_{S_{\alpha\beta}} g_{\alpha\beta}(\mathbf{u}) \mathbf{n}_\beta \cdot \text{Tr}_\beta^0(\mathbf{v}_\beta) dS \right), \quad (32)$$

and $g_{\alpha\beta}(\mathbf{u}) = (d_{\alpha\beta} - u_{\alpha n} - u_{\beta n})^-$.

Taking into account the following inequality for real numbers:

$$\left(\sum_{i=1}^m c_i\right)^2 \leq m \sum_{i=1}^m c_i^2, \quad c_i \in \mathbb{R}, \quad i = 1, 2, \dots, m, \quad m \in \mathbb{N}, \quad (33)$$

and Schwarz inequality, we obtain

$$\left(\int_{S_{\alpha\beta}} g_{\alpha\beta}(\mathbf{u}) \mathbf{n}_\alpha \cdot \text{Tr}_\alpha^0(\mathbf{v}_\alpha) dS\right)^2 \leq 3 q_{\alpha\beta}(\mathbf{u}) \|\text{Tr}_\alpha^0(\mathbf{v}_\alpha)\|_{[L_2(S_{\alpha\beta})]^m}^2, \quad (34)$$

where $q_{\alpha\beta}(\mathbf{u}) = \|g_{\alpha\beta}(\mathbf{u}) \mathbf{n}_\alpha\|_{[L_2(S_{\alpha\beta})]^3}^2 + \varepsilon_{\alpha\beta}$, $\varepsilon_{\alpha\beta} > 0$, $\beta \in B_\alpha$, $\alpha = 1, 2, \dots, N$.

From trace theorems in [20], it follows, that

$$(\exists T_{\alpha\beta} > 0) (\forall \mathbf{v}_\alpha \in V_\alpha^0) \left\{ \|\text{Tr}_\alpha^0(\mathbf{v}_\alpha)\|_{[L_2(S_{\alpha\beta})]^3}^2 \leq T_{\alpha\beta}^2 \|\mathbf{v}_\alpha\|_{V_\alpha}^2 \right\}. \quad (35)$$

Substituting (35) into (34), we come to inequality $\left|\int_{S_{\alpha\beta}} g_{\alpha\beta}(\mathbf{u}) \mathbf{n}_\alpha \cdot \text{Tr}_\alpha^0(\mathbf{v}_\alpha) dS\right| \leq \tilde{s}_{\alpha\beta}(\mathbf{u}) \|\mathbf{v}_\alpha\|_{V_\alpha}$, where $\tilde{s}_{\alpha\beta}(\mathbf{u}) = T_{\alpha\beta} \sqrt{3q_{\alpha\beta}(\mathbf{u})} > 0$, $\beta \in B_\alpha$, $\alpha = 1, 2, \dots, N$.

Similarly to this, we obtain the same inequality for the second term of relationship (32). Hence

$$|j'_{\alpha\beta}(\mathbf{u}, \mathbf{v})| \leq \frac{1}{\theta} \left(\tilde{s}_{\alpha\beta}(\mathbf{u}) \|\mathbf{v}_\alpha\|_{V_\alpha} + \tilde{s}_{\beta\alpha}(\mathbf{u}) \|\mathbf{v}_\beta\|_{V_\beta} \right).$$

As a result, we find

$$|J'_\theta(\mathbf{u}, \mathbf{v})| \leq \sum_{\{\alpha, \beta\} \in Q} |j'_{\alpha\beta}(\mathbf{u}, \mathbf{v})| \leq \tilde{R}(\mathbf{u}) \|\mathbf{v}\|_{V_0},$$

where $\tilde{R}(\mathbf{u}) = \frac{1}{\theta} \sum_{\{\alpha, \beta\} \in Q} (\tilde{s}_{\alpha\beta}(\mathbf{u}) + \tilde{s}_{\beta\alpha}(\mathbf{u})) > 0$. Inequality (29) is proved.

Now, we prove that condition (30) holds. For this we use the next inequality

$$(\forall y, z \in \mathbb{R}) \left\{ [(y - z)^- - y^-] z \leq 0 \right\}, \quad (36)$$

Rewrite $J'_\theta(\mathbf{u} + \mathbf{w}, \mathbf{v}) - J'_\theta(\mathbf{u}, \mathbf{v})$ in the following way:

$$J'_\theta(\mathbf{u} + \mathbf{w}, \mathbf{v}) - J'_\theta(\mathbf{u}, \mathbf{v}) = -\frac{1}{\theta} \sum_{\{\alpha, \beta\} \in Q} h_{\alpha\beta}(\mathbf{u}, \mathbf{w}, \mathbf{v}), \quad (37)$$

where $h_{\alpha\beta}(\mathbf{u}, \mathbf{w}, \mathbf{v}) = \int_{S_{\alpha\beta}} r_{\alpha\beta}(\mathbf{u}, \mathbf{w}) (v_{\alpha n} + v_{\beta n}) dS$, $r_{\alpha\beta}(\mathbf{u}, \mathbf{w}) = g_{\alpha\beta}(\mathbf{u} + \mathbf{w}) - g_{\alpha\beta}(\mathbf{u})$, $\mathbf{x} \in S_{\alpha\beta}$.

In view of property (36), we obtain

$$(\forall \mathbf{u}, \mathbf{v} \in V_0) \{ r_{\alpha\beta}(\mathbf{u}, \mathbf{v}) (v_{\alpha n} + v_{\beta n}) \leq 0, \quad \mathbf{x} \in S_{\alpha\beta} \}.$$

Therefore, since $\theta > 0$, we come to inequality

$$J'_\theta(\mathbf{u} + \mathbf{v}, \mathbf{v}) - J'_\theta(\mathbf{u}, \mathbf{v}) = -\frac{1}{\theta} \sum_{\{\alpha, \beta\} \in Q} \int_{S_{\alpha\beta}} r_{\alpha\beta}(\mathbf{u}, \mathbf{v}) (v_{\alpha n} + v_{\beta n}) dS \geq 0, \quad \forall \mathbf{u}, \mathbf{v} \in V_0.$$

Finally, let us prove the satisfaction of property (31). Using the next inequality for real numbers

$$(\forall y, z \in \mathbb{R}) \{ |y^- - z^-| \leq |y - z| \}, \quad (38)$$

we obtain

$$r_{\alpha\beta}(\mathbf{u}, \mathbf{w}) \leq |w_{\alpha n} + w_{\beta n}|, \quad \mathbf{x} \in S_{\alpha\beta}. \quad (39)$$

Taking into account, that the components of outer unit normal to $\partial\Omega_\alpha$ satisfy property $\max_{j=1,2,3} |n_{\alpha j}| \leq 1$, and using inequalities (33) and (39), we find

$$\int_{S_{\alpha\beta}} r_{\alpha\beta}^2(\mathbf{u}, \mathbf{w}) dS \leq 6 \left(\|\text{Tr}_\alpha^0(\mathbf{w}_\alpha)\|_{[L_2(S_{\alpha\beta})]^3}^2 + \|\text{Tr}_\beta^0(\mathbf{w}_\beta)\|_{[L_2(S_{\alpha\beta})]^3}^2 \right). \quad (40)$$

Now, let us write $h_{\alpha\beta}(\mathbf{u}, \mathbf{w}, \mathbf{v})$ as follows:

$$h_{\alpha\beta}(\mathbf{u}, \mathbf{w}, \mathbf{v}) = \int_{S_{\alpha\beta}} r_{\alpha\beta}(\mathbf{u}, \mathbf{w}) v_{\alpha n} dS + \int_{S_{\alpha\beta}} r_{\alpha\beta}(\mathbf{u}, \mathbf{w}) v_{\beta n} dS.$$

Consider the term $\int_{S_{\alpha\beta}} r_{\alpha\beta}(\mathbf{u}, \mathbf{w}) v_{\alpha n} dS$ in more detail. Let us use inequalities (33), (40), and Schwarz inequality:

$$\begin{aligned} \left(\int_{S_{\alpha\beta}} r_{\alpha\beta}(\mathbf{u}, \mathbf{w}) v_{\alpha n} dS \right)^2 &\leq \int_{S_{\alpha\beta}} r_{\alpha\beta}^2(\mathbf{u}, \mathbf{w}) dS \int_{S_{\alpha\beta}} v_{\alpha n}^2 dS \leq \\ &\leq 18 \left(\|\text{Tr}_\alpha^0(\mathbf{w}_\alpha)\|_{[L_2(S_{\alpha\beta})]^3}^2 + \|\text{Tr}_\beta^0(\mathbf{w}_\beta)\|_{[L_2(S_{\alpha\beta})]^3}^2 \right) \|\text{Tr}_\alpha^0(\mathbf{v}_\alpha)\|_{[L_2(S_{\alpha\beta})]^3}^2. \end{aligned}$$

Using inequality (35), we find further

$$\left| \int_{S_{\alpha\beta}} r_{\alpha\beta}(\mathbf{u}, \mathbf{w}) v_{\alpha n} dS \right| \leq T_{\alpha\beta 1} \|\mathbf{v}_\alpha\|_{V_\alpha} \left(\|\mathbf{w}_\alpha\|_{V_\alpha} + \|\mathbf{w}_\beta\|_{V_\beta} \right),$$

where $T_{\alpha\beta 1} = 3 T_{\alpha\beta} T_{\alpha\beta}^* \sqrt{2} > 0$, $T_{\alpha\beta}^* = \max \{T_{\alpha\beta}, T_{\beta\alpha}\} > 0$. In much the same way we come to inequality

$$\left| \int_{S_{\alpha\beta}} r_{\alpha\beta}(\mathbf{u}, \mathbf{w}) v_{\beta n} dS \right| \leq T_{\alpha\beta 2} \|\mathbf{v}_\beta\|_{V_\beta} \left(\|\mathbf{w}_\alpha\|_{V_\alpha} + \|\mathbf{w}_\beta\|_{V_\beta} \right),$$

where $T_{\alpha\beta 2} = 3 T_{\beta\alpha} T_{\alpha\beta}^* \sqrt{2} > 0$.

Taking into account the last two inequalities, we establish

$$\begin{aligned} |h_{\alpha\beta}(\mathbf{u}, \mathbf{w}, \mathbf{v})| &\leq \left| \int_{S_{\alpha\beta}} r_{\alpha\beta}(\mathbf{u}, \mathbf{w}) v_{\alpha n} dS \right| + \left| \int_{S_{\alpha\beta}} r_{\alpha\beta}(\mathbf{u}, \mathbf{w}) v_{\beta n} dS \right| \leq \\ &\leq C_{\alpha\beta} \left(\|\mathbf{v}_\alpha\|_{V_\alpha} \|\mathbf{w}_\alpha\|_{V_\alpha} + \|\mathbf{v}_\alpha\|_{V_\alpha} \|\mathbf{w}_\beta\|_{V_\beta} + \|\mathbf{v}_\beta\|_{V_\beta} \|\mathbf{w}_\alpha\|_{V_\alpha} + \|\mathbf{v}_\beta\|_{V_\beta} \|\mathbf{w}_\beta\|_{V_\beta} \right), \end{aligned}$$

where $C_{\alpha\beta} = \max \{T_{\alpha\beta 1}, T_{\alpha\beta 2}\} > 0$.

As a result, we obtain

$$|J'_\theta(\mathbf{u} + \mathbf{w}, \mathbf{v}) - J'_\theta(\mathbf{u}, \mathbf{v})| \leq \frac{1}{\theta} \sum_{\alpha, \beta=1}^N |h_{\alpha\beta}(\mathbf{u}, \mathbf{w}, \mathbf{v})| \leq$$

$$\begin{aligned}
&\leq \frac{2C}{\theta} \left(N \sum_{\alpha=1}^N \|\mathbf{v}_\alpha\|_{V_\alpha} \|\mathbf{w}_\alpha\|_{V_\alpha} + \sum_{\alpha=1}^N \|\mathbf{v}_\alpha\|_{V_\alpha} \sum_{\beta=1}^N \|\mathbf{w}_\beta\|_{V_\beta} \right) \leq \\
&\leq \frac{2C(N+1)}{\theta} \sum_{\alpha=1}^N \|\mathbf{v}_\alpha\|_{V_\alpha} \sum_{\beta=1}^N \|\mathbf{w}_\beta\|_{V_\beta} \leq D \|\mathbf{v}\| \|\mathbf{w}\|, \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V_0,
\end{aligned}$$

where $D = 2C(N+1)/\theta > 0$, $C = \max_{1 \leq \alpha, \beta \leq N} C_{\alpha\beta} > 0$. \square

Theorem 2. Suppose that V_0 is closed reflexive Banach space, $J_\theta(\mathbf{u})$ is Gateaux differentiable, and conditions (19) – (22), (27), (29) – (31) hold. Then there exist a unique solution of nonquadratic minimization problem (26) in V_0 , and this problem is equivalent to the following nonlinear variational equation:

$$F'_\theta(\mathbf{u}, \mathbf{v}) = A(\mathbf{u}, \mathbf{v}) + J'_\theta(\mathbf{u}, \mathbf{v}) - L(\mathbf{v}) = 0, \quad \forall \mathbf{v} \in V_0, \quad \mathbf{u} \in V_0. \quad (41)$$

Proof. As shown in [17], due to the properties (19) – (22), the functional $F(\mathbf{u})$ is strictly convex, and coercive ($\lim_{\|\mathbf{u}\|_{V_0} \rightarrow \infty} F(\mathbf{u}) = \infty$), and the differential $F'(\mathbf{u}, \mathbf{v})$ is linear and continuous in \mathbf{v} .

From the property (30), it follows that the penalty term $J_\theta(\mathbf{u})$ is convex in V_0 [20].

Now consider the properties of functional

$$F_\theta(\mathbf{u}) = F(\mathbf{u}) + J_\theta(\mathbf{u}), \quad \mathbf{u} \in V_0.$$

This functional is Gateaux differentiable in V_0 :

$$F'_\theta(\mathbf{u}, \mathbf{v}) = F'(\mathbf{u}, \mathbf{v}) + J'_\theta(\mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in V_0,$$

and strictly convex, as the sum of convex functional $J_\theta(\mathbf{u})$ and strictly convex functional $F(\mathbf{u})$. In addition, since functionals $F'(\mathbf{u}, \mathbf{v})$ and $J'_\theta(\mathbf{u}, \mathbf{v})$ are linear and continuous in \mathbf{v} , it follows that $F'_\theta(\mathbf{u}, \mathbf{v})$ is also linear and continuous in \mathbf{v} . Hence, according to [24], functional $F_\theta(\mathbf{u})$ is weakly lower semicontinuous. Due to the coercivity of $F(\mathbf{u})$ and property (26), we obtain that $\lim_{\|\mathbf{u}\|_{V_0} \rightarrow \infty} F_\theta(\mathbf{u}) = \infty$.

Since V_0 is closed reflexive Banach space, $F_\theta(\mathbf{u})$ is weakly lower semicontinuous and coercive, then according to the theorem of existence of the minimum in reflexive Banach space [24], there exists a solution of minimization problem (26) in the space V_0 .

Furthermore, the functional $F_\theta(\mathbf{u})$ is strictly convex and Gateaux differentiable in V_0 . Therefore, according to the theorem about necessary and sufficient conditions of the minimum in reflexive Banach space [24], the solution of problem (26) is unique, and this problem is equivalent to the variational equation (41). \square

Now let us prove that the solution of penalty variational equation (41) converges strongly to the solution of original variational inequality (24) as $\theta \rightarrow 0$.

Let us rewrite variational equation (41) in the following equivalent form

$$F'_\theta(\mathbf{u}, \mathbf{v}) = A(\mathbf{u}, \mathbf{v}) - \langle L, \mathbf{v} \rangle + \frac{1}{\theta} \langle \Phi(\mathbf{u}), \mathbf{v} \rangle = 0, \quad \forall \mathbf{v} \in V_0, \quad \mathbf{u} \in V_0, \quad (42)$$

where $\langle Y, \mathbf{u} \rangle = Y(\mathbf{u})$ is the action of a functional $Y \in V_0^*$ on an element $\mathbf{u} \in V_0$, V_0^* is the space dual to V_0 , $\Phi = \Psi' : V_0 \rightarrow V_0^*$ is Gateaux derivative of functional $\Psi(\mathbf{u}) = \theta J_\theta(\mathbf{u})$, and $\langle \Phi(\mathbf{u}), \mathbf{v} \rangle = \langle \Psi'(\mathbf{u}), \mathbf{v} \rangle = \theta J'_\theta(\mathbf{u}, \mathbf{v})$.

We have proved the next lemma.

Lemma 3. *Suppose that properties (30) and (31) hold. Then operator $\Phi : V_0 \rightarrow V_0^*$ in problem (42) is a penalty operator for the kinematically allowable displacements set K , i.e.*

1). Φ is monotone in V_0 :

$$(\forall \mathbf{u}, \mathbf{v} \in V_0) \{ \langle \Phi(\mathbf{u}) - \Phi(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle \geq 0 \};$$

2). Φ satisfies the Lipschitz condition in V_0 :

$$(\exists C > 0) (\forall \mathbf{u}, \mathbf{v} \in V_0) \left\{ \|\Phi(\mathbf{u}) - \Phi(\mathbf{v})\|_{V_0^*} \leq C \|\mathbf{u} - \mathbf{v}\|_{V_0} \right\};$$

3). The kernel of operator Φ is equal to set K :

$$\text{Ker}(\Phi) = \{ \mathbf{u} : \mathbf{u} \in V_0, \Phi(\mathbf{u}) = 0 \} = K.$$

Proof. The monotonicity of operator Φ follows from the condition (30), and the satisfaction of Lipschitz condition follows from the property (31).

If $\mathbf{u} \in K$, then $\Phi(\mathbf{u}) \equiv 0$, and, on the contrary, if $\Phi(\mathbf{u}) \equiv 0$, we have $\mathbf{u} \in K$. Hence, $\text{Ker}(\Phi) = K$. For more details see [17, 19]. \square

Now, using the results of works [25, 26, 27], let us prove the proposition about the convergence of penalty method, applied to the variational inequality (24).

Theorem 3. *Suppose that $K \subset V_0$ is a convex closed subset of Hilbert space V_0 , bilinear form $A(\mathbf{u}, \mathbf{v})$, $\mathbf{u}, \mathbf{v} \in V_0$ and linear form $L(\mathbf{v})$, $\mathbf{v} \in V_0$ satisfy properties (19) – (22), $\Phi : V_0 \rightarrow V_0^*$ is penalty operator for set K , $\bar{\mathbf{u}} \in K$ is a unique solution of variational inequality (24), and $\bar{\mathbf{u}}_\theta \in V_0$ is a unique solution of penalty variational equation (42) with penalty parameter $\theta > 0$. Then $\bar{\mathbf{u}}_\theta \xrightarrow{\theta \rightarrow 0} \bar{\mathbf{u}}$ strongly in V_0 , i.e. $\|\bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}\|_{V_0} \xrightarrow{\theta \rightarrow 0} 0$.*

Proof. In works [25, 26] it is proved that, if conditions (20) – (22) hold, Φ is a penalty operator for set K and there exist solutions of problems (24) and (42), then the sequence $\{\bar{\mathbf{u}}_\theta\}$ is bounded:

$$(\exists \tilde{C} \in (0; \infty)) (\forall \theta > 0) \left\{ \|\bar{\mathbf{u}}_\theta\|_{V_0} \leq \tilde{C} \right\},$$

and there exists such subsequence $\{\bar{\mathbf{u}}_{\theta_1}\} \subset \{\bar{\mathbf{u}}_\theta\}$, which converges weakly in V_0 to some solution of variational inequality (24), i.e.

$$(\exists \{\bar{\mathbf{u}}_{\theta_1}\} \subset \{\bar{\mathbf{u}}_\theta\}) (\forall Y \in V_0^*) \left\{ \langle Y, \bar{\mathbf{u}}_{\theta_1} \rangle \xrightarrow{\theta_1 \rightarrow 0} \langle Y, \bar{\mathbf{u}} \rangle \right\}.$$

Moreover, in [25, 26] it is shown that any weakly convergent subsequence of sequence $\{\bar{\mathbf{u}}_\theta\}$ converges weakly in V_0 to some solution of variational problem (24).

Now let us assume that variational problems (24) and (42) have unique solutions. Then, as follows from above, the sequence $\{\bar{\mathbf{u}}_\theta\}$ has a unique partial weak limit $\bar{\mathbf{u}} \in K$.

Since the sequence $\{\bar{\mathbf{u}}_\theta\}$ has a unique weak limit point, and is bounded, then according to theorem in [24], it is weakly convergent to this point, i.e.

$$(\forall Y \in V_0^*) \left\{ \langle Y, \bar{\mathbf{u}}_\theta \rangle \xrightarrow{\theta \rightarrow 0} \langle Y, \bar{\mathbf{u}} \rangle \right\}, \quad (43)$$

where $\bar{\mathbf{u}} \in K$ is the unique solution of variational inequality (24).

Further, let us show that $\{\bar{\mathbf{u}}_\theta\}$ converges strongly to $\bar{\mathbf{u}} \in K$ as $\theta \rightarrow 0$.

Due to (43), we get

$$\langle L, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}} \rangle \xrightarrow{\theta \rightarrow 0} 0, \quad A(\mathbf{v}, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}) = \langle A'_1(\mathbf{v}), \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}} \rangle \xrightarrow{\theta \rightarrow 0} 0, \quad \forall \mathbf{v} \in V_0, \quad (44)$$

where $A'_1(\mathbf{v})$ is the Gateaux derivative of functional $A_1(\mathbf{v}) = \frac{1}{2}A(\mathbf{v}, \mathbf{v})$, $\mathbf{v} \in V_0$.

Since $\bar{\mathbf{u}}_\theta$ is a solution of penalty variational equation (42), it is obvious that

$$\langle L, \bar{\mathbf{u}}_\theta - \mathbf{v} \rangle = A(\bar{\mathbf{u}}_\theta, \bar{\mathbf{u}}_\theta - \mathbf{v}) + \frac{1}{\theta} \langle \Phi(\bar{\mathbf{u}}_\theta), \bar{\mathbf{u}}_\theta - \mathbf{v} \rangle, \quad \forall \mathbf{v} \in K.$$

Taking into account the monotonicity of penalty operator Φ and the property $(\forall \mathbf{v} \in K) \{\Phi(\mathbf{v}) = 0\}$, we obtain

$$\langle L, \bar{\mathbf{u}}_\theta - \mathbf{v} \rangle = A(\bar{\mathbf{u}}_\theta, \bar{\mathbf{u}}_\theta - \mathbf{v}) + \frac{1}{\theta} \langle \Phi(\bar{\mathbf{u}}_\theta) - \Phi(\mathbf{v}), \bar{\mathbf{u}}_\theta - \mathbf{v} \rangle \geq A(\bar{\mathbf{u}}_\theta, \bar{\mathbf{u}}_\theta - \mathbf{v}), \quad \forall \mathbf{v} \in K.$$

In view of this property and the nonnegativity of bilinear form A , we get the inequality $A(\bar{\mathbf{u}}, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}) - \langle L, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}} \rangle \leq A(\bar{\mathbf{u}}_\theta, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}) - \langle L, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}} \rangle \leq 0$. Hence

$$A(\bar{\mathbf{u}}, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}) \leq A(\bar{\mathbf{u}}_\theta, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}) \leq \langle L, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}} \rangle. \quad (45)$$

Passing to the limit in expression (45) as $\theta \rightarrow 0$, and taking into account property (44), we obtain

$$A(\bar{\mathbf{u}}_\theta, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}) \xrightarrow{\theta \rightarrow 0} 0.$$

Further, in view of coercivity of bilinear form A , it follows that

$$0 \leq B \|\bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}\|_{V_0}^2 \leq A(\bar{\mathbf{u}}_\theta, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}) - A(\bar{\mathbf{u}}, \bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}) \xrightarrow{\theta \rightarrow 0} 0, \quad B > 0.$$

As a result, we establish that $\|\bar{\mathbf{u}}_\theta - \bar{\mathbf{u}}\|_{V_0} \xrightarrow{\theta \rightarrow 0} 0$. \square

Thus, using the penalty method we have reduced the solution of original variational inequality (24) to the solution of the nonlinear variational equation in the whole space V_0 , which depends on the penalty parameter $\theta > 0$. We also have proved the existence of the unique solution of penalty variational equation (42) and its strong convergence to the solution of initial variational inequality (24) as penalty parameter θ tends to zero.

5 Iterative methods for nonlinear variation equations

Consider an abstract nonlinear in \mathbf{u} variational equation in form (41), where V_0 is closed reflexive Banach space, $A(\mathbf{u}, \mathbf{v})$ is bilinear form, set in $V_0 \times V_0$,

$L(\mathbf{v})$ is linear functional, and the term $J'_\theta(\mathbf{u}, \mathbf{v})$ is linear in \mathbf{v} and nonlinear in \mathbf{u} . Suppose that conditions (19) – (22), (29) – (31) are satisfied. Hence, there exists a unique solution of the problem (41).

For the numerical solution of the nonlinear variational equation (41) let us use the following iterative method [16, 17, 18, 19]:

$$G(\mathbf{u}^{k+1}, \mathbf{v}) = G(\mathbf{u}^k, \mathbf{v}) - \gamma [A(\mathbf{u}^k, \mathbf{v}) + J'_\theta(\mathbf{u}^k, \mathbf{v}) - L(\mathbf{v})], \quad k = 0, 1, \dots, \quad (46)$$

where $G(\mathbf{u}, \mathbf{v})$ is some bilinear form assigned in $V_0 \times V_0$, $\mathbf{u}^k \in V_0$, $k = 1, 2, \dots$ is the k -th approximation to the exact solution $\bar{\mathbf{u}} \in V_0$ of the problem (41), $\mathbf{u}^0 \in V_0$ is an initial guess, and $\gamma \in \mathbb{R}$ is an iterative parameter.

We have proved the following proposition about the convergence of the iterative method (46).

Theorem 4. *Suppose that bilinear form $G(\mathbf{u}, \mathbf{v})$ is symmetric, continuous and coercive:*

$$(\forall \mathbf{u}, \mathbf{v} \in V_0) \{G(\mathbf{u}, \mathbf{v}) = G(\mathbf{v}, \mathbf{u})\}, \quad (47)$$

$$(\exists \tilde{M} > 0) (\forall \mathbf{u}, \mathbf{v} \in V_0) \left\{ |G(\mathbf{u}, \mathbf{v})| \leq \tilde{M} \|\mathbf{u}\|_{V_0} \|\mathbf{v}\|_{V_0} \right\}, \quad (48)$$

$$(\exists \tilde{B} > 0) (\forall \mathbf{u} \in V_0) \left\{ G(\mathbf{u}, \mathbf{u}) \geq \tilde{B} \|\mathbf{u}\|_{V_0}^2 \right\}, \quad (49)$$

properties (20) – (22), (29) – (31) are satisfied, and the iteration parameter lies in the interval $\gamma \in (0; \gamma_2)$, $\gamma_2 = 2B\tilde{B}/M_*^2$, $M_* = M + D$.

Then the sequence $\{\mathbf{u}^k\}$, obtained by iterative method (46) converges strongly in V_0 to the exact solution $\bar{\mathbf{u}} \in V_0$ of the variational equation (41), i.e. $\|\mathbf{u}^k - \bar{\mathbf{u}}\|_{V_0} \xrightarrow{k \rightarrow \infty} 0$, and the convergence rate in the energy norm $\|\mathbf{u}\|_G = \sqrt{G(\mathbf{u}, \mathbf{u})}$ is linear:

$$\|\mathbf{u}^{k+1} - \bar{\mathbf{u}}\|_G \leq q \|\mathbf{u}^k - \bar{\mathbf{u}}\|_G, \quad q = \sqrt{1 - \gamma (2B - \gamma M_*^2 / \tilde{B})} / \tilde{M} < 1. \quad (50)$$

Moreover, the maximal convergence rate reaches as $\gamma = \bar{\gamma} = B\tilde{B}/M_*^2$.

Proof. Since bilinear form $G(\mathbf{u}, \mathbf{v})$ is symmetric, continuous and coercive, we may introduce a scalar product and norm

$$(\mathbf{u}, \mathbf{v})_G = G(\mathbf{u}, \mathbf{v}), \quad \|\mathbf{u}\|_G = \sqrt{G(\mathbf{u}, \mathbf{u})}, \quad \mathbf{u}, \mathbf{v} \in V_0.$$

From properties (48) and (49), it follows that norms $\|\cdot\|_G$ and $\|\cdot\|_{V_0}$ are equivalent in space V_0 .

In each step $k \in \{0, 1, \dots\}$ of method (46), we have to solve the linear variational problem:

$$G(\mathbf{u}, \mathbf{v}) = Y^k(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_0, \quad \mathbf{u} \in \mathbf{V}_0, \quad (51)$$

where $Y^k(\mathbf{v}) = G(\mathbf{u}^k, \mathbf{v}) - \gamma [A(\mathbf{u}^k, \mathbf{v}) + J'_\theta(\mathbf{u}^k, \mathbf{v}) - L(\mathbf{v})]$ is linear in \mathbf{v} , $\mathbf{u}^k \in V_0$. Using properties (20), (22), (29) and (48), we obtain that $Y^k(\mathbf{v})$ is continuous:

$$(\exists Z_k > 0) (\forall \mathbf{v} \in V_0) \left\{ |Y^k(\mathbf{v})| \leq Z_k \|\mathbf{v}\|_{V_0} \right\}, \quad (52)$$

where $Z_k = \tilde{M} \|\mathbf{u}^k\|_{V_0} + |\gamma| (M \|\mathbf{u}^k\|_{V_0} + \tilde{R}(\mathbf{u}^k) + T) + \varepsilon > 0$, $\varepsilon > 0$.

Since conditions (47) – (49) and (52) are satisfied, we see that the problem (51) has a unique solution $\mathbf{u} = \mathbf{u}^{k+1} \in V_0$.

Now let us show, that the sequence of solutions of problems (51) converges strongly to the solution of initial variational equation (41).

Suppose that $\bar{\mathbf{u}} \in V_0$ is the exact solution of problem (41). Introduce a notation $\boldsymbol{\varphi}^k := \mathbf{u}^k - \bar{\mathbf{u}} \in V_0$, $k = 0, 1, \dots$, and rewrite (46) as follows:

$$G(\bar{\mathbf{u}} + \boldsymbol{\varphi}^{k+1}, \mathbf{v}) = G(\bar{\mathbf{u}} + \boldsymbol{\varphi}^k, \mathbf{v}) - \gamma [A(\bar{\mathbf{u}} + \boldsymbol{\varphi}^k, \mathbf{v}) + J'_\theta(\bar{\mathbf{u}} + \boldsymbol{\varphi}^k, \mathbf{v}) - L(\mathbf{v})].$$

Subtracting from this expression the identity $G(\bar{\mathbf{u}}, \mathbf{v}) \equiv G(\bar{\mathbf{u}}, \mathbf{v}) - \gamma [A(\bar{\mathbf{u}}, \mathbf{v}) + J'_\theta(\bar{\mathbf{u}}, \mathbf{v}) - L(\mathbf{v})]$, we obtain

$$G(\boldsymbol{\varphi}^{k+1}, \mathbf{v}) = G(\boldsymbol{\varphi}^k, \mathbf{v}) - \gamma [A(\boldsymbol{\varphi}^k, \mathbf{v}) + J'_\theta(\bar{\mathbf{u}} + \boldsymbol{\varphi}^k, \mathbf{v}) - J'_\theta(\bar{\mathbf{u}}, \mathbf{v})]. \quad (53)$$

Let us define a functional $H_\theta(\mathbf{u}, \mathbf{w}, \mathbf{v}) = J'_\theta(\mathbf{u} + \mathbf{w}, \mathbf{v}) - J'_\theta(\mathbf{u}, \mathbf{v})$, $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V_0$, which is linear in \mathbf{v} .

Due to properties (30) and (31), the following conditions hold:

$$(\forall \mathbf{u}, \mathbf{v} \in V_0) \{ H_\theta(\mathbf{u}, \mathbf{v}, \mathbf{v}) \geq 0 \}, \quad (54)$$

$$(\exists D > 0) (\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V_0) \{ |H_\theta(\mathbf{u}, \mathbf{w}, \mathbf{v})| \leq D \|\mathbf{w}\|_{V_0} \|\mathbf{v}\|_{V_0} \}. \quad (55)$$

Let us rewrite expression (53) in the form

$$G(\boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k, \mathbf{v}) = -\gamma [A(\boldsymbol{\varphi}^k, \mathbf{v}) + H_\theta(\bar{\mathbf{u}}, \boldsymbol{\varphi}^k, \mathbf{v})], \quad \forall \mathbf{v} \in V_0. \quad (56)$$

If we take $\mathbf{v} := \boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k$ in (56), we shall have

$$\|\boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k\|_G^2 \leq |\gamma| (|A(\boldsymbol{\varphi}^k, \boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k)| + |H_\theta(\bar{\mathbf{u}}, \boldsymbol{\varphi}^k, \boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k)|).$$

Taking into account the continuity of bilinear form (20), and property (55), we get

$$\|\boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k\|_G^2 \leq |\gamma| M_* \|\boldsymbol{\varphi}^k\|_{V_0} \|\boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k\|_{V_0}, \quad M_* = M + D > 0.$$

Further, in view of relation between norms

$$\|\boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k\|_{V_0} \leq \|\boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k\|_G / \sqrt{\bar{B}}, \quad (57)$$

we come to inequalities

$$\sqrt{\bar{B}} \|\boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k\|_{V_0} \leq \|\boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k\|_G \leq \frac{|\gamma| M_*}{\sqrt{\bar{B}}} \|\boldsymbol{\varphi}^k\|_{V_0}. \quad (58)$$

Now let us take $\mathbf{v} := \boldsymbol{\varphi}^{k+1} + \boldsymbol{\varphi}^k$ in expression (56). Then $G(\boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k, \boldsymbol{\varphi}^{k+1} + \boldsymbol{\varphi}^k) = -\gamma [A(\boldsymbol{\varphi}^k, \boldsymbol{\varphi}^{k+1} + \boldsymbol{\varphi}^k) + H_\theta(\bar{\mathbf{u}}, \boldsymbol{\varphi}^k, \boldsymbol{\varphi}^{k+1} + \boldsymbol{\varphi}^k)]$. This relation can be written as

$$\begin{aligned} \|\boldsymbol{\varphi}^k\|_G^2 - \|\boldsymbol{\varphi}^{k+1}\|_G^2 &= \gamma [2A(\boldsymbol{\varphi}^k, \boldsymbol{\varphi}^k) + 2H_\theta(\bar{\mathbf{u}}, \boldsymbol{\varphi}^k, \boldsymbol{\varphi}^k) + \\ &+ A(\boldsymbol{\varphi}^k, \boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k) + H_\theta(\bar{\mathbf{u}}, \boldsymbol{\varphi}^k, \boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k)]. \end{aligned}$$

In view of properties (20) and (55), we come to inequality

$$A(\boldsymbol{\varphi}^k, \boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k) + H_\theta(\bar{\mathbf{u}}, \boldsymbol{\varphi}^k, \boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k) \geq -M_* \|\boldsymbol{\varphi}^k\|_{V_0} \|\boldsymbol{\varphi}^{k+1} - \boldsymbol{\varphi}^k\|_{V_0}.$$

Suppose that $\gamma \geq 0$. Then, taking into account the coercitivity of bilinear form (21), property (54), and the previous inequality, we obtain

$$\|\varphi^k\|_G^2 - \|\varphi^{k+1}\|_G^2 \geq \gamma \left[2B \|\varphi^k\|_{V_0}^2 - M_* \|\varphi^k\|_{V_0} \|\varphi^{k+1} - \varphi^k\|_{V_0} \right].$$

In view of inequalities (57) and (58), we find further

$$\|\varphi^k\|_G^2 - \|\varphi^{k+1}\|_G^2 \geq \frac{\gamma}{M} \left(2B - \gamma M_*^2 / \tilde{B} \right) \|\varphi^k\|_G^2. \quad (59)$$

If the following inequality

$$\gamma \left(2B - \gamma M_*^2 / \tilde{B} \right) > 0 \quad (60)$$

holds, then the sequence $\|\varphi^k\|_G^2$ will be monotonically nonincreasing: $\|\varphi^k\|_G^2 \geq \|\varphi^{k+1}\|_G^2$, and, hence, $\|\varphi^k\|_G^2 \xrightarrow{k \rightarrow \infty} \omega$, where $\omega \geq 0$. Passing to the limit as $k \rightarrow \infty$ in expression (59), we obtain $0 \geq \frac{\gamma}{M} \left(2B - \gamma M_*^2 / \tilde{B} \right) \omega$, i.e. $\omega = 0$, and, therefore $\|\varphi^k\|_G \xrightarrow{k \rightarrow \infty} 0$. From inequality (60) we establish the interval of allowable values of iteration parameter γ :

$$\gamma \in (0; \gamma_2), \quad \gamma_2 = 2B\tilde{B} / M_*^2.$$

Since the norms $\|\cdot\|_G$ and $\|\cdot\|_{V_0}$ are equivalent, we have $\|\varphi^k\|_{V_0} \xrightarrow{k \rightarrow \infty} 0$, and, hence, $\|\mathbf{u}^k - \bar{\mathbf{u}}\|_{V_0} \xrightarrow{k \rightarrow \infty} 0$.

Using inequality (59), we find estimate

$$\|\varphi^{k+1}\|_G^2 \leq q^2 \|\varphi^k\|_G^2, \quad q^2(\gamma) = 1 - \frac{2B}{M} \gamma + \frac{M_*^2}{M\tilde{B}} \gamma^2. \quad (61)$$

It is not hard to show, that $q^2 \in (0; 1)$ for $\gamma \in (0; \gamma_2)$. We obtain this from the next relations:

$$q^2(0) = q^2(\gamma_2) = 1, \quad \bar{\gamma} = \arg \min_{\gamma \in (0; \gamma_2)} q^2(\gamma) = \gamma_2 / 2 = B\tilde{B} / M_*^2.$$

As follows from estimate (61), the convergence rate is maximal if the parameter q is minimal, i.e., if $\gamma = \bar{\gamma}$. \square

Remark 1. Suppose that term $J'_\theta(\mathbf{u}, \mathbf{v})$ is Gateaux differentiable in \mathbf{u} . Then conditions (29), (30) in **theorem 4** can be replaced by the following properties

$$(\forall \mathbf{u}, \mathbf{v} \in V_0) \{ J''_\theta(\mathbf{u}, \mathbf{v}, \mathbf{v}) \geq 0 \}, \quad (62)$$

$$(\exists D > 0) (\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V_0) \{ |J''_\theta(\mathbf{u}, \mathbf{v}, \mathbf{w})| \leq D \|\mathbf{v}\|_{V_0} \|\mathbf{w}\|_{V_0} \}. \quad (63)$$

Proof. Let us apply to $J''_\theta(\mathbf{u}, \mathbf{v}, \mathbf{w})$ the Lagrange formula of finite increments [24]:

$$(\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V_0) (\exists \tau \in (0; 1)) \{ J''_\theta(\mathbf{u} + \tau \mathbf{w}, \mathbf{v}, \mathbf{v}) = J'_\theta(\mathbf{u} + \mathbf{w}, \mathbf{v}) - J'_\theta(\mathbf{u}, \mathbf{v}) \}.$$

Then the satisfaction of conditions (62) and (63) yields the satisfaction of properties (30) and (31). \square

Now let us investigate the stability of iterative method (46) to errors.

Suppose that conditions of **theorem 4** are satisfied. Then for any $\mathbf{u}^k \in V_0$, $k = 0, 1, \dots$ there exists a unique solution $\mathbf{u} = \mathbf{u}^{k+1} \in V_0$ of problem (51).

Therefore, there exists an operator $R : \mathbf{u}^k \in V_0 \rightarrow \mathbf{u}^{k+1} \in V_0$, which maps every element $\mathbf{u}^k \in V_0$ to the solution $\mathbf{u} = \mathbf{u}^{k+1}$ of the problem (51), and the iterative method (46) can be written in the following form

$$\mathbf{u}^{k+1} = R(\mathbf{u}^k), \quad k = 0, 1, \dots \quad (64)$$

Now assume that in each step k of iterative method (64) we get some computational errors. Then this iterative method will take the form:

$$\check{\mathbf{u}}^0 = \mathbf{u}^0 + \boldsymbol{\varepsilon}^0, \quad (65)$$

$$\check{\mathbf{u}}^{k+1} = R(\check{\mathbf{u}}^k) + \boldsymbol{\varepsilon}^{k+1}, \quad k = 0, 1, \dots, \quad (66)$$

where $\check{\mathbf{u}}^{k+1}$, $k = 0, 1, \dots$ is an approximate solution of problem (51), $\boldsymbol{\varepsilon}^{k+1}$, $k = 0, 1, \dots$ is a computational error, which occurs in each step k , and $\boldsymbol{\varepsilon}^0$ is an initial guess error.

Corollary 1. *Suppose that conditions of **theorem 4** are satisfied, and errors which occur in each step k of iterative method (46) are uniformly bounded, i.e.*

$$(\exists \varepsilon > 0) (\forall k \in \{0, 1, \dots\}) \{ \|\boldsymbol{\varepsilon}^k\|_G \leq \varepsilon \}.$$

Then the following estimates hold:

$$\|\check{\mathbf{u}}^k - \bar{\mathbf{u}}\|_G \leq q \|\check{\mathbf{u}}^{k-1} - \bar{\mathbf{u}}\|_G + \varepsilon, \quad (67)$$

$$\|\check{\mathbf{u}}^k - \bar{\mathbf{u}}\|_G \leq q^k \|\mathbf{u}^0 - \bar{\mathbf{u}}\|_G + \frac{\varepsilon}{1-q}, \quad (68)$$

$$\|\check{\mathbf{u}}^k - \bar{\mathbf{u}}\|_G \leq \frac{q}{1-q} \|\check{\mathbf{u}}^k - \check{\mathbf{u}}^{k-1}\|_G + \frac{\varepsilon}{1-q}, \quad (69)$$

$$\|\check{\mathbf{u}}^k - \mathbf{u}^k\|_G \leq 2q^k \|\mathbf{u}^0 - \bar{\mathbf{u}}\|_G + \frac{\varepsilon}{1-q}, \quad k = 1, 2, \dots, \quad (70)$$

where $\bar{\mathbf{u}} \in V_0$ is the exact solution of problem (41).

The proof of this proposition follows from property (50).

Thus, the errors, which occur in each step of iterative method (46), do not accumulate.

Now consider the nonstationary iterative method for the solution of nonlinear variational equation (41), where bilinear forms $G(\mathbf{u}, \mathbf{v})$ are different in each iteration [19, 28].

In space $V_0 \times V_0$ introduce a sequence of bilinear forms $\{G^k : V_0 \times V_0 \rightarrow \mathbb{R}\}$, $k = 0, 1, \dots$, which satisfy the property

$$(\forall Y \in V_0^*) (\exists! \bar{\mathbf{u}} \in V_0) (\forall \mathbf{v} \in V_0) \{G^k(\bar{\mathbf{u}}, \mathbf{v}) - Y(\mathbf{v}) \equiv 0\}.$$

For the solution of nonlinear variational equation (41), we have proposed the following nonstationary iterative method [19, 28]:

$$G^k(\mathbf{u}^{k+1}, \mathbf{v}) = G^k(\mathbf{u}^k, \mathbf{v}) - \gamma [A(\mathbf{u}^k, \mathbf{v}) + J'_\theta(\mathbf{u}^k, \mathbf{v}) - L(\mathbf{v})], \quad k = 0, 1, \dots, \quad (71)$$

where $\mathbf{u}^k \in V_0$ is the k -th approximation to the exact solution of problem (41), and $\gamma \in \mathbb{R}$ is an iterative parameter.

We also have proved the next proposition about the convergence of this method.

Theorem 5. *Suppose that conditions (20) – (22), (29) – (31) hold, bilinear forms $G^k(\mathbf{u}, \mathbf{v})$ satisfy properties*

$$(\forall \mathbf{u}, \mathbf{v} \in V_0) \{G^k(\mathbf{u}, \mathbf{v}) = G^k(\mathbf{v}, \mathbf{u})\}, \quad (72)$$

$$(\exists \tilde{M} > 0) (\forall k \in \{0, 1, \dots\}) (\forall \mathbf{u}, \mathbf{v} \in V_0) \left\{ |G^k(\mathbf{u}, \mathbf{v})| \leq \tilde{M} \|\mathbf{u}\|_{V_0} \|\mathbf{v}\|_{V_0} \right\}, \quad (73)$$

$$(\exists \tilde{B} > 0) (\forall k \in \{0, 1, \dots\}) (\forall \mathbf{u} \in V_0) \left\{ G^k(\mathbf{u}, \mathbf{u}) \geq \tilde{B} \|\mathbf{u}\|_{V_0}^2 \right\}, \quad (74)$$

$$(\exists k_0 \in \{0, 1, \dots\}) (\forall k \geq k_0) (\forall \mathbf{u} \in V_0) \left\{ G^k(\mathbf{u}, \mathbf{u}) \geq G^{k+1}(\mathbf{u}, \mathbf{u}) \right\}, \quad (75)$$

and iterative parameter γ lies in the interval $\gamma \in (0; \gamma_2)$, $\gamma_2 = 2B\tilde{B}/(M + D)^2$.

Then the sequence $\{\mathbf{u}^k\}$ obtained by the nonstationary iterative method (71) converges strongly in V_0 to the exact solution $\bar{\mathbf{u}} \in V_0$ of the variational equation (41), i.e. $\|\mathbf{u}^k - \bar{\mathbf{u}}\|_{V_0} \xrightarrow{k \rightarrow \infty} 0$.

The proof of this theorem is similar to the proof of **theorem 4**. We omit the details.

Note, that in iterative methods (46) and (71) we can also take the parameter γ differently in each iteration. The purpose of the nonstationary choice of γ might be the improvement of the convergence rate of iterative methods (46) and (71).

6 Parallel domain decomposition schemes

Note, that in the most general case iterative methods (46) and (71) applied to solve nonlinear penalty variational equations (41) for multibody contact problems do not lead to domain decomposition. Therefore, we now consider such variants of these methods, which lead to domain decomposition, namely, which reduce the solution of the original multibody contact problem in Ω to the solution of a sequence of separate linear variational problems in subdomains Ω_α , $\alpha = 1, 2, \dots, N$.

Let us take the bilinear form G in the iterative method (46) as follows [18, 19]:

$$G(\mathbf{u}, \mathbf{v}) = A(\mathbf{u}, \mathbf{v}) + X(\mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in V_0, \quad (76)$$

where $X(\mathbf{u}, \mathbf{v}) : V_0 \times V_0 \rightarrow \mathbb{R}$ is the next bilinear form [18, 19]:

$$X(\mathbf{u}, \mathbf{v}) = \frac{1}{\theta} \sum_{\{\alpha, \beta\} \in Q} \int_{S_{\alpha\beta}} (u_{\alpha n} v_{\alpha n} \psi_{\alpha\beta} + u_{\beta n} v_{\beta n} \psi_{\beta\alpha}) dS, \quad \mathbf{u}, \mathbf{v} \in V_0. \quad (77)$$

Here $\psi_{\alpha\beta}(\mathbf{x}) = \left\{ 0, \mathbf{x} \in S_{\alpha\beta} \setminus S_{\alpha\beta}^1 \right\} \vee \left\{ 1, \mathbf{x} \in S_{\alpha\beta}^1 \right\}$ are characteristic functions of some given subareas $S_{\alpha\beta}^1 \subseteq S_{\alpha\beta}$ of possible contact zones $S_{\alpha\beta}$, $\alpha = 1, 2, \dots, N$, $\beta \in B_\alpha$.

The iterative method (46) with bilinear form (77) can be written in the following equivalent way:

$$A(\tilde{\mathbf{u}}^{k+1}, \mathbf{v}) + X(\tilde{\mathbf{u}}^{k+1}, \mathbf{v}) = L(\mathbf{v}) + X(\mathbf{u}^k, \mathbf{v}) - J'_\theta(\mathbf{u}^k, \mathbf{v}), \quad \forall \mathbf{v} \in V_0, \quad (78)$$

$$\mathbf{u}^{k+1} = \gamma \tilde{\mathbf{u}}^{k+1} + (1 - \gamma) \mathbf{u}^k, \quad k = 0, 1, \dots \quad (79)$$

Lemma 4. *Suppose that surfaces $S_{\alpha\beta}$, $\{\alpha, \beta\} \in Q$ are piecewise smooth. Then bilinear form (77) is symmetric, continuous and nonnegative, i.e.*

$$(\forall \mathbf{u}, \mathbf{v} \in V_0) \{X(\mathbf{u}, \mathbf{v}) = X(\mathbf{v}, \mathbf{u})\}, \quad (80)$$

$$(\exists Z > 0) (\forall \mathbf{u}, \mathbf{v} \in V_0) \{|X(\mathbf{u}, \mathbf{v})| \leq Z \|\mathbf{u}\|_{V_0} \|\mathbf{v}\|_{V_0}\}, \quad (81)$$

$$(\forall \mathbf{u} \in V_0) \{X(\mathbf{u}, \mathbf{u}) \geq 0\}. \quad (82)$$

Proof. It is obvious, that conditions (80) and (82) hold. Thus, let us show the continuity of bilinear form (77).

We can write $X(\mathbf{u}, \mathbf{v}) = \sum_{\{\alpha, \beta\} \in Q} X_{\alpha\beta}(\mathbf{u}, \mathbf{v})$, where

$$X_{\alpha\beta}(\mathbf{u}, \mathbf{v}) = \frac{1}{\theta} \left(\int_{S_{\alpha\beta}} \psi_{\alpha\beta} u_{\alpha n} v_{\alpha n} dS + \int_{S_{\alpha\beta}} \psi_{\beta\alpha} u_{\beta n} v_{\beta n} dS \right), \quad \{\alpha, \beta\} \in Q.$$

The first term can be written in the following form:

$$\int_{S_{\alpha\beta}} \psi_{\alpha\beta} u_{\alpha n} v_{\alpha n} dS = \int_{S_{\alpha\beta}} \psi_{\alpha\beta} [\mathbf{n}_\alpha \cdot \text{Tr}_\alpha^0(\mathbf{u}_\alpha)] [\mathbf{n}_\alpha \cdot \text{Tr}_\alpha^0(\mathbf{v}_\alpha)] dS,$$

Taking into account, that the functions $\psi_{\alpha\beta}$ and the components of unit normals \mathbf{n}_α are bounded, and using inequality (33) and Schwarz inequality, we obtain

$$\left(\int_{S_{\alpha\beta}} \psi_{\alpha\beta} u_{\alpha n} v_{\alpha n} dS \right)^2 \leq 9 \|Tr_\alpha^0(\mathbf{u}_\alpha)\|_{[L_2(S_{\alpha\beta})]^3}^2 \|Tr_\alpha^0(\mathbf{v}_\alpha)\|_{[L_2(S_{\alpha\beta})]^3}^2.$$

In view of inequality (35), we find further

$$\left| \int_{S_{\alpha\beta}} \psi_{\alpha\beta} u_{\alpha n} v_{\alpha n} dS \right| \leq 3 \tilde{W}_{\alpha\beta} \|\mathbf{u}_\alpha\|_{V_\alpha} \|\mathbf{v}_\alpha\|_{V_\alpha}, \quad \tilde{W}_{\alpha\beta} = T_{\alpha\beta}^2 + \tilde{\varepsilon}_{\alpha\beta} > 0, \quad \tilde{\varepsilon}_{\alpha\beta} > 0.$$

Similar inequality can be obtained for the second term of $X_{\alpha\beta}$. Thus

$$|X_{\alpha\beta}(\mathbf{u}, \mathbf{v})| \leq \frac{3 W_{\alpha\beta}}{\theta} \left(\|\mathbf{u}_\alpha\|_{V_\alpha} \|\mathbf{v}_\alpha\|_{V_\alpha} + \|\mathbf{u}_\beta\|_{V_\beta} \|\mathbf{v}_\beta\|_{V_\beta} \right), \quad W_{\alpha\beta} = \max \{ \tilde{W}_{\alpha\beta}, \tilde{W}_{\beta\alpha} \}.$$

As a result, we establish

$$|X(\mathbf{u}, \mathbf{v})| \leq \frac{3 W}{\theta} \sum_{\alpha=1}^N \sum_{\beta=1}^N \left(\|\mathbf{u}_\alpha\|_{V_\alpha} \|\mathbf{v}_\alpha\|_{V_\alpha} + \|\mathbf{u}_\beta\|_{V_\beta} \|\mathbf{v}_\beta\|_{V_\beta} \right) =$$

$$= \frac{6WN}{\theta} \sum_{\alpha=1}^N \|\mathbf{u}_\alpha\|_{V_\alpha} \|\mathbf{v}_\alpha\|_{V_\alpha} \leq Z \|\mathbf{u}\|_{V_0} \|\mathbf{v}\|_{V_0}, \quad \forall \mathbf{u}, \mathbf{v} \in V_0,$$

where $Z = 6WN/\theta > 0$, $W = \max_{1 \leq \alpha, \beta \leq N} W_{\alpha\beta} > 0$. \square

From this **lemma** and **lemma 1**, it follows that bilinear form (76) is symmetric, continuous and coercive with constants $\tilde{M} = M + Z$ and $\tilde{B} = B$ respectively. In addition, due to **lemmas 1, 2**, and **4**, functionals $L(\mathbf{v})$, $X(\mathbf{u}^k, \mathbf{v})$, and $J'_\theta(\mathbf{u}^k, \mathbf{v})$ are linear and continuous in \mathbf{v} . Therefore, there exists a unique solution $\mathbf{u} = \tilde{\mathbf{u}}^{k+1} \in V_0$ of variational problem (78).

Thus, the conditions of **theorem 4** are satisfied, and we obtain the next proposition.

Theorem 6. *Suppose that $\gamma \in (0; \gamma_2)$, $\gamma_2 = 2B^2/(M + D)^2$. Then the sequence $\{\mathbf{u}^k\}$, obtained by iterative method (78) – (79), which is equivalent to iterative method (46) with bilinear form (76), converges strongly in V_0 to the exact solution $\tilde{\mathbf{u}} \in V_0$ of nonlinear penalty variational equation (41) for unilateral multibody contact problem, i.e. $\|\mathbf{u}^k - \tilde{\mathbf{u}}\|_{V_0} \xrightarrow{k \rightarrow \infty} 0$. Moreover, the convergence rate in norm $\|\cdot\|_G$ is linear (50), where $q = \sqrt{1 - \gamma(2B - \gamma M_*^2/B)/(M + Z)}$, and the maximal rate reaches as $\gamma = \bar{\gamma} = B^2/M_*^2$, $M_* = M + D$.*

Now let us show that iterative method (78) – (79) leads to domain decomposition.

Due to relationships

$$J'_\theta(\mathbf{u}, \mathbf{v}) = -\frac{1}{\theta} \sum_{\alpha=1}^N \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} (d_{\alpha\beta} - u_{\alpha n} - u_{\beta n})^- v_{\alpha n} dS,$$

$$X(\mathbf{u}, \mathbf{v}) = \frac{1}{\theta} \sum_{\alpha=1}^N \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} \psi_{\alpha\beta} u_{\alpha n} v_{\alpha n} dS,$$

method (78) – (79) rewrites as follows:

$$\sum_{\alpha=1}^N a_\alpha(\tilde{\mathbf{u}}_\alpha^{k+1}, \mathbf{v}_\alpha) + \frac{1}{\theta} \sum_{\alpha=1}^N \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} \psi_{\alpha\beta} (\tilde{u}_{\alpha n}^{k+1} - u_{\alpha n}^k) v_{\alpha n} dS =$$

$$= \sum_{\alpha=1}^N l_\alpha(\mathbf{v}_\alpha) + \frac{1}{\theta} \sum_{\alpha=1}^N \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} (d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k)^- v_{\alpha n} dS, \quad (83)$$

$$\mathbf{u}_\alpha^{k+1} = \gamma \tilde{\mathbf{u}}_\alpha^{k+1} + (1 - \gamma) \mathbf{u}_\alpha^k, \quad \alpha = 1, 2, \dots, N, \quad k = 0, 1, \dots \quad (84)$$

Since the common quantities of the subdomains are known from the previous iteration, the variational equation (83) splits into N variational equations in separate subdomains Ω_α . Therefore, method (83) – (84) can be written in the following equivalent form:

$$a_\alpha(\tilde{\mathbf{u}}_\alpha^{k+1}, \mathbf{v}_\alpha) + \frac{1}{\theta} \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} \psi_{\alpha\beta} \tilde{u}_{\alpha n}^{k+1} v_{\alpha n} dS =$$

$$\begin{aligned}
&= l_\alpha(\mathbf{v}_\alpha) + \frac{1}{\theta} \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} \psi_{\alpha\beta} u_{\alpha n}^k v_{\alpha n} dS + \\
&+ \frac{1}{\theta} \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} (d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k)^- v_{\alpha n} dS, \quad \forall \mathbf{v}_\alpha \in V_\alpha^0, \quad (85)
\end{aligned}$$

$$\mathbf{u}_\alpha^{k+1} = \gamma \tilde{\mathbf{u}}_\alpha^{k+1} + (1 - \gamma) \mathbf{u}_\alpha^k, \quad \alpha = 1, 2, \dots, N, \quad k = 0, 1, \dots \quad (86)$$

Since bilinear forms $a_\alpha(\mathbf{u}_\alpha, \mathbf{v}_\alpha)$, $X_\alpha(\mathbf{u}_\alpha, \mathbf{v}_\alpha) = \frac{1}{\theta} \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} \psi_{\alpha\beta} u_{\alpha n} v_{\alpha n} dS$ are symmetric, continuous and coercive, and functionals $l_\alpha(\mathbf{v}_\alpha)$, $X_\alpha(\mathbf{u}_\alpha^k, \mathbf{v}_\alpha)$, $\frac{1}{\theta} \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} (d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k)^- v_{\alpha n} dS$ are linear and continuous in \mathbf{v}_α , it follows that there exists a unique solution $\mathbf{u}_\alpha^* = \tilde{\mathbf{u}}_\alpha^{k+1} \in V_\alpha^0$ of each variational equation (85). Furthermore, it is obvious to see that the unique solution \mathbf{u}^* of variational equation (78), i.e. (83), takes the form $\mathbf{u}^* = \tilde{\mathbf{u}}^{k+1} = (\mathbf{u}_1^*, \mathbf{u}_2^*, \dots, \mathbf{u}_N^*)^T \in V_0$. Therefore, the solution of variational equation (78) is equivalent to the solution of N variational equations (85) in separate subdomains, and iterative processes (78) – (79) and (85) – (86) are equivalent.

Now, consider iterative method (85) – (86) in more detail.

In each iteration of this method we have to solve in parallel N variational equations (85), which correspond to some elasticity problems in subdomains with prescribed Robin boundary conditions on possible contact areas:

$$\tilde{\sigma}_{\alpha\beta n}^{k+1} + \psi_{\alpha\beta} \tilde{u}_{\alpha n}^{k+1} / \theta = (d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k)^- / \theta + \psi_{\alpha\beta} u_{\alpha n}^k / \theta \quad \text{on } S_{\alpha\beta}. \quad (87)$$

Here $\tilde{\sigma}_{\alpha\beta n}^{k+1}$ are unknown normal stresses on possible contact areas $S_{\alpha\beta}$.

Therefore, iterative method (85) – (86) refers to parallel Robin–Robin type domain decomposition schemes.

Since domain decomposition method (85) – (86) and iterative method (78) – (79) are equivalent, the convergence **theorem 6** also holds for the method (85) – (86).

Taking different characteristic functions $\psi_{\alpha\beta}$ in (85), i.e. different subareas $S_{\alpha\beta}^1$ of possible contact zones $S_{\alpha\beta}$, we can obtain different particular cases of domain decomposition method (85) – (86).

Thus, taking $\psi_{\alpha\beta}(\mathbf{x}) \equiv 1$, $\forall \alpha, \beta$, i.e. $S_{\alpha\beta}^1 = S_{\alpha\beta}$, we get domain decomposition scheme with Robin boundary conditions on whole possible contact areas:

$$\tilde{\sigma}_{\alpha\beta n}^{k+1} + \tilde{u}_{\alpha n}^{k+1} / \theta = (d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k)^- / \theta + u_{\alpha n}^k / \theta \quad \text{on } S_{\alpha\beta}.$$

Therefore, we have named this domain decomposition method as full parallel Robin–Robin domain decomposition scheme [18, 19].

Taking $\psi_{\alpha\beta}(\mathbf{x}) \equiv 0$, $\forall \alpha, \beta$, i.e. $S_{\alpha\beta}^1 = \emptyset$, we get parallel Neumann–Neumann domain decomposition scheme [16, 17, 18, 19]:

$$a_\alpha(\tilde{\mathbf{u}}_\alpha^{k+1}, \mathbf{v}_\alpha) = l_\alpha(\mathbf{v}_\alpha) + \frac{1}{\theta} \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} (d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k)^- v_{\alpha n} dS, \quad \forall \mathbf{v}_\alpha \in V_\alpha^0, \quad (88)$$

$$\mathbf{u}_\alpha^{k+1} = \gamma \tilde{\mathbf{u}}_\alpha^{k+1} + (1 - \gamma) \mathbf{u}_\alpha^k, \quad \alpha = 1, 2, \dots, N, \quad k = 0, 1, \dots \quad (89)$$

In each step k of this scheme we have to solve in parallel N variational equations (88), which correspond to elasticity problems in subdomains with Neumann boundary conditions on possible contact areas:

$$\tilde{\sigma}_{\alpha\beta n}^{k+1} = \sigma_{\alpha\beta n}^k = (d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k)^- / \theta \quad \text{on } S_{\alpha\beta}.$$

Note, that in the most general case, we can choose functions $\psi_{\alpha\beta}$, i.e. surfaces $S_{\alpha\beta}^1$, differently for each α, β .

Moreover, we can choose functions $\psi_{\alpha\beta}$ differently at each iteration k , i.e.

$$\psi_{\alpha\beta}(\mathbf{x}) = \psi_{\alpha\beta}^k(\mathbf{x}) = \{0, \mathbf{x} \in S_{\alpha\beta} \setminus S_{\alpha\beta}^k\} \vee \{1, \mathbf{x} \in S_{\alpha\beta}^k\}, \quad (90)$$

where $S_{\alpha\beta}^k \subseteq S_{\alpha\beta}$, $k = 0, 1, \dots$ are some given subareas of possible contact zones $S_{\alpha\beta}$, $\alpha = 1, 2, \dots, N$, $\beta \in B_\alpha$.

As a result we obtain following nonstationary Robin–Robin type domain decomposition scheme

$$\begin{aligned} a_\alpha(\tilde{\mathbf{u}}_\alpha^{k+1}, \mathbf{v}_\alpha) + \frac{1}{\theta} \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} \psi_{\alpha\beta}^k (\tilde{u}_{\alpha n}^{k+1} - u_{\alpha n}^k) v_{\alpha n} dS = \\ = l_\alpha(\mathbf{v}_\alpha) + \frac{1}{\theta} \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} (d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k)^- v_{\alpha n} dS, \quad \forall \mathbf{v}_\alpha \in V_\alpha^0, \end{aligned} \quad (91)$$

$$\mathbf{u}_\alpha^{k+1} = \gamma \tilde{\mathbf{u}}_\alpha^{k+1} + (1 - \gamma) \mathbf{u}_\alpha^k, \quad \alpha = 1, 2, \dots, N, \quad k = 0, 1, \dots \quad (92)$$

This domain decomposition scheme is equivalent to nonstationary iterative method (71) with bilinear forms

$$G^k(\mathbf{u}, \mathbf{v}) = A(\mathbf{u}, \mathbf{v}) + X^k(\mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in V_0, \quad k = 0, 1, \dots, \quad (93)$$

where

$$X^k(\mathbf{u}, \mathbf{v}) = \frac{1}{\theta} \sum_{\{\alpha, \beta\} \in Q} \int_{S_{\alpha\beta}} (u_{\alpha n} v_{\alpha n} \psi_{\alpha\beta}^k + u_{\beta n} v_{\beta n} \psi_{\beta\alpha}^k) dS, \quad \mathbf{u}, \mathbf{v} \in V_0, \quad k = 0, 1, \dots \quad (94)$$

Bilinear forms $X^k(\mathbf{u}, \mathbf{v})$ are symmetric, nonnegative, and continuous with constant $Z > 0$. Hence, bilinear forms (93) satisfy properties (72) – (74), where $\tilde{M} = M + Z$, $\tilde{B} = B$.

It is obvious to see that condition (75) in **theorem 5** for bilinear forms (93) is equivalent to the following condition

$$(\exists k_0 \in \{0, 1, \dots\}) (\forall k \geq k_0) \{X^k(\mathbf{u}, \mathbf{u}) \geq X^{k+1}(\mathbf{u}, \mathbf{u})\},$$

which by-turn is equivalent to the condition

$$(\exists k_0 \in \{0, 1, \dots\}) (\forall k \geq k_0) (\forall \alpha) (\forall \beta \in B_\alpha) (\forall \mathbf{x} \in S_{\alpha\beta}) \left\{ \psi_{\alpha\beta}^k(\mathbf{x}) \geq \psi_{\alpha\beta}^{k+1}(\mathbf{x}) \right\}. \quad (95)$$

Therefore, from **theorem 5** we obtain the next proposition about convergence of nonstationary domain decomposition scheme (91) – (92).

Theorem 7. Suppose that $\gamma \in (0; \gamma_2)$, $\gamma_2 = 2B^2/(M + D)^2$, and functions $\psi_{\alpha\beta}^k$ satisfy property (95). Then the sequence $\{\mathbf{u}^k\}$, obtained by nonstationary

domain decomposition scheme (91) – (92), converges strongly in V_0 to the exact solution of nonlinear penalty variational equation (41) for unilateral multibody contact problem, i.e. $\|\mathbf{u}^k - \bar{\mathbf{u}}\|_{V_0} \xrightarrow{k \rightarrow \infty} 0$.

Now let us consider a particular case of domain decomposition method (91) – (92). In each iteration k let us choose functions $\psi_{\alpha\beta}^k$ as follows [16, 18, 19, 29]:

$$\psi_{\alpha\beta}^k(\mathbf{x}) = \chi_{\alpha\beta}^k(\mathbf{x}) = \begin{cases} 0, & d_{\alpha\beta}(\mathbf{x}) - u_{\alpha n}^k(\mathbf{x}) - u_{\beta n}^k(\mathbf{x}') \geq 0 \\ 1, & d_{\alpha\beta}(\mathbf{x}) - u_{\alpha n}^k(\mathbf{x}) - u_{\beta n}^k(\mathbf{x}') < 0 \end{cases}, \quad (96)$$

where $\mathbf{x} \in S_{\alpha\beta}$, $\mathbf{x}' = P(\mathbf{x}) \in S_{\beta\alpha}$. Then, taking into consideration that $(d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k)^- = (d_{\alpha\beta} - u_{\alpha n}^k - u_{\beta n}^k) \chi_{\alpha\beta}^k$, we obtain the method [16, 18, 19, 29]:

$$a_\alpha(\tilde{\mathbf{u}}_\alpha^{k+1}, \mathbf{v}_\alpha) + \frac{1}{\theta} \sum_{\beta \in B_\alpha} \int_{S_{\alpha\beta}} \chi_{\alpha\beta}^k (\tilde{u}_{\alpha n}^{k+1} - (d_{\alpha\beta} - u_{\beta n}^k)) v_{\alpha n} dS = l_\alpha(\mathbf{v}_\alpha), \quad (97)$$

$$\mathbf{u}_\alpha^{k+1} = \gamma \tilde{\mathbf{u}}_\alpha^{k+1} + (1 - \gamma) \mathbf{u}_\alpha^k, \quad \alpha = 1, 2, \dots, N, \quad k = 0, 1, \dots \quad (98)$$

In each step k of this method we have to solve in parallel N variational equations (97), which correspond to elasticity problems in subdomains with prescribed displacements $d_{\alpha\beta} - u_{\beta n}^k$ through penalty on some subareas of possible contact zones $S_{\alpha\beta}$. Therefore, we can conventionally name this method as nonstationary parallel Dirichlet–Dirichlet domain decomposition scheme.

Note, that in some particular cases of penalty Robin–Robin domain decomposition method (91) – (92) (for example $\psi_{\alpha\beta}^k(\mathbf{x}) \equiv 1, \forall \alpha, \beta$ or $\psi_{\alpha\beta}^k(\mathbf{x}) \equiv \chi_{\alpha\beta}^k(\mathbf{x}), \forall \alpha, \beta$) we can pass the limit to $\theta \rightarrow 0$, and obtain domain decomposition schemes without penalty [19, 30].

The advantages of proposed domain decomposition schemes are their simplicity, and the regularization of the original contact problem because of the use of penalty term. These domain decomposition schemes have only one iteration loop, which deals with domain decomposition and nonlinearity of unilateral contact conditions.

The disadvantage of proposed methods is that we have to choose a penalty parameter.

Finally, let us say that iterative methods (46) and (71) for the solution of nonlinear variational equations are rather general. From these methods, besides parallel Robin–Robin type domain decomposition schemes (85) – (86) and (91) – (92), we also can obtain other different particular iterative methods for the solution of penalty variational equation of unilateral multibody contact problems, which do not lead to domain decomposition.

Thus, taking bilinear form $G(\mathbf{u}, \mathbf{v}) = A(\mathbf{u}, \mathbf{v}) + \tilde{X}(\mathbf{u}, \mathbf{v})$, $\tilde{X}(\mathbf{u}, \mathbf{v}) = \frac{1}{\theta} \sum_{\{\alpha, \beta\} \in Q} \int_{S_{\alpha\beta}} (u_{\alpha n} + u_{\beta n}) (v_{\alpha n} + v_{\beta n}) dS$, and iterative parameter $\gamma = 1$ in (46), we obtain iterative method for the solution of multibody unilateral contact problems, which can be viewed as a generalization of penalty iteration method, proposed in [27] for the solution of crack problems with nonpenetration conditions.

Taking in method (71) bilinear forms G^k in each step k as follows

$$G^k(\mathbf{u}, \mathbf{v}) = A(\mathbf{u}, \mathbf{v}) + \tilde{X}^k(\mathbf{u}, \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in V_0,$$

where

$$\tilde{X}^k(\mathbf{u}, \mathbf{v}) = \frac{1}{\theta} \sum_{\{\alpha, \beta\} \in Q} \int_{S_{\alpha\beta}} (u_{\alpha n} + u_{\beta n}) (v_{\alpha n} + v_{\beta n}) \chi_{\alpha\beta}^k dS, \quad \mathbf{u}, \mathbf{v} \in V_0,$$

$$\chi_{\alpha\beta}^k(\mathbf{x}) = \begin{cases} 0, & d_{\alpha\beta}(\mathbf{x}) - u_{\alpha n}^k(\mathbf{x}) - u_{\beta n}^k(\mathbf{x}') \geq 0 \\ 1, & d_{\alpha\beta}(\mathbf{x}) - u_{\alpha n}^k(\mathbf{x}) - u_{\beta n}^k(\mathbf{x}') < 0 \end{cases}, \quad \mathbf{x} \in S_{\alpha\beta}, \quad \mathbf{x}' = P(\mathbf{x}) \in S_{\beta\alpha},$$

and iterative parameter $\gamma = 1$, we obtain iterative method, which can be viewed as active set method, i.e. semi-smooth Newton method, for unilateral multibody contact problems. For one-body contact problems active set method was introduced in [20]. The convergence theorem for active set method, as a variant of semi-smooth Newton method, was proved in [31].

Note, that bilinear forms $\tilde{X}(\mathbf{u}, \mathbf{v})$ and $\tilde{X}^k(\mathbf{u}, \mathbf{v})$ are symmetric, nonnegative and coercive.

However, these two particular cases of methods (46) and (71) do not lead to the domain decomposition.

In the next section we investigate numerical efficiency of proposed by us penalty parallel Robin–Robin type domain decomposition schemes (85) – (86) and (91) – (92).

7 Numerical investigations

We provide numerical investigation of proposed domain decomposition schemes for plane unilateral contact problems of two elastic bodies $\Omega_\alpha \subset \mathbb{R}^2$, $\alpha = 1, 2$. For the numerical solution of linear variational problems in subdomains, we use finite element method (FEM) with linear and quadratic triangular elements.

At first let us compare the convergence rates of different particular domain decomposition schemes.

Consider contact problem of two transversally isotropic elastic bodies Ω_α , $\alpha = 1, 2$ with the plane of isotropy, parallel to the plane $x_2 = 0$ (fig. 2).

The material properties of the bodies are: $E_\alpha/E'_\alpha = 2$, $G_\alpha/G'_\alpha = 2$, $\nu_\alpha = \nu'_\alpha = 0.3$, where E_α , ν_α , and G_α – are the elasticity modulus, Poisson's ratio, and the shear modulus for the body Ω_α in the plane of isotropy, and E'_α , ν'_α , G'_α are these constants in the orthogonal direction, $\alpha = 1, 2$.

The length and height of each body is the same, and is equal to $4b$. The distance between bodies before the deformation is $d_{12}(\mathbf{x}) = r x_1^2/b^2$, the compression of the bodies is $\Delta \approx 2.154434 r$, $r = 10^{-3}b$, and the possible contact area is $S_{12} = \{\mathbf{x} = (x_1, x_2)^T : x_1 \in [0; 2b], x_2 = 4b\}$.

The problem is solved by parallel Robin–Robin domain decomposition schemes, using FEM with 1595 quadratic triangular elements in each body (15 elements on each side of possible contact area S_{12}).

We take the penalty parameter in the form $\theta = 4bc(1/E'_1 + 1/E'_2)$, $c = 0.05$, where c is dimensionless penalty coefficient. The initial guesses for displacements $u_{\alpha n}^0$, $\alpha = 1, 2$ are chosen using the bar model [19]:

$$u_{1n}^0(\mathbf{x}) = \frac{4bc[d_{12}(\mathbf{x}) - \Delta]^-}{\theta E'_1(1+c)}, \quad \mathbf{x} \in S_{12}, \quad u_{2n}^0(\mathbf{x}) = \frac{4bc[d_{12}(\mathbf{x}) - \Delta]^-}{\theta E'_2(1+c)} + \Delta, \quad \mathbf{x} \in S_{21}.$$

We use the following stopping criterion for domain decomposition schemes

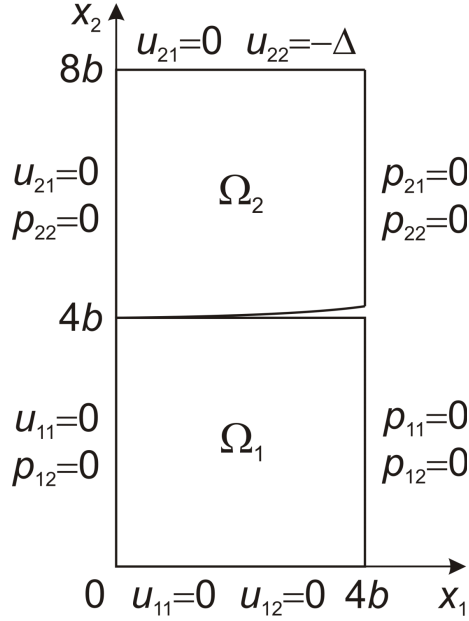


Fig. 2. Unilateral contact of two transversally-isotropic bodies

$$\|u_{\alpha n}^{k+1} - u_{\alpha n}^k\|_2 / \|u_{\alpha n}^{k+1}\|_2 \leq \varepsilon_u, \quad \alpha = 1, 2, \dots, N, \quad (99)$$

where $\|u_{\alpha n}\|_2 = \sqrt{\sum_j [u_{\alpha n}(\mathbf{x}^j)]^2}$ is the discrete norm, $\mathbf{x}^j \in S_{12}$ are the finite element nodes on the possible contact area, and $\varepsilon_u > 0$ is the relative accuracy for displacements.

Fig. 3 shows the approximations to dimensionless normal contact stress $\sigma_n^*(x_1, x_2) = \sigma_{12n}(x_1, x_2) / |\sigma_{12n}(0, x_2)|$, $x_2 = 4b$, $(x_1, x_2)^T \in S_{12}$, obtained by parallel Neumann–Neumann scheme (88) – (89) ($\psi_{12}(\mathbf{x}) = \psi_{21}(\mathbf{x}) \equiv 0$) at iterations $k = 1, 2, 4, 21$ (curves 1–4) for optimal iterative parameter $\bar{\gamma} = 0.173$ and accuracy $\varepsilon_u = 10^{-3}$. The dashed curve represents the exact solution for two half-spaces, obtained in [32]. Hence, the real contact area is $S_{12}^* \approx \llbracket 0; b \rrbracket$, where $\llbracket y; z \rrbracket = \left\{ \mathbf{x} = (x_1, x_2)^T : x_1 \in [y; z], x_2 = 4b \right\}$.

At fig. 4 and fig. 5 the convergence rates of different particular domain decomposition schemes are compared.

The total iteration number m dependence on iteration parameter γ for accuracy $\varepsilon_u = 10^{-3}$ is shown at fig. 4, and its dependence on logarithmic accuracy $\lg \varepsilon_u$ for optimal iteration parameter $\gamma = \bar{\gamma}$ is shown at fig. 5.

The first curve at these figures represents the parallel Neumann–Neumann scheme ($S_{12}^1 = S_{21}^1 = \emptyset$, $\psi_{12}(\mathbf{x}) = \psi_{21}(\mathbf{x}) \equiv 0$), curves 2, 3, 4 and 5 correspond to the parallel Robin–Robin schemes (85) – (86) with $S_{12}^1 = S_{21}^1$ equal to $\llbracket 0; 0.5 \rrbracket$, $\llbracket 0; 1 \rrbracket$, $\llbracket 0; 1.5 \rrbracket$, and $\llbracket 0; 2 \rrbracket$ ($S_{12}^1 = S_{21}^1 = S_{12}$, $\psi_{12}(\mathbf{x}) = \psi_{21}(\mathbf{x}) \equiv 1$) respectively. Curve 3 also represents the nonstationary parallel Dirichlet–Dirichlet scheme (97) – (98).

The optimal iterative parameters $\bar{\gamma}$ for the schemes represented at curves 1–5 are $\bar{\gamma} = 0.173, 0.39, 0.72, 0.85$ and 0.92 respectively. For $\gamma = \bar{\gamma}$ and accuracy $\varepsilon_u = 10^{-3}$ these schemes converge in 21, 11, 5, 11 and 14 iterations.

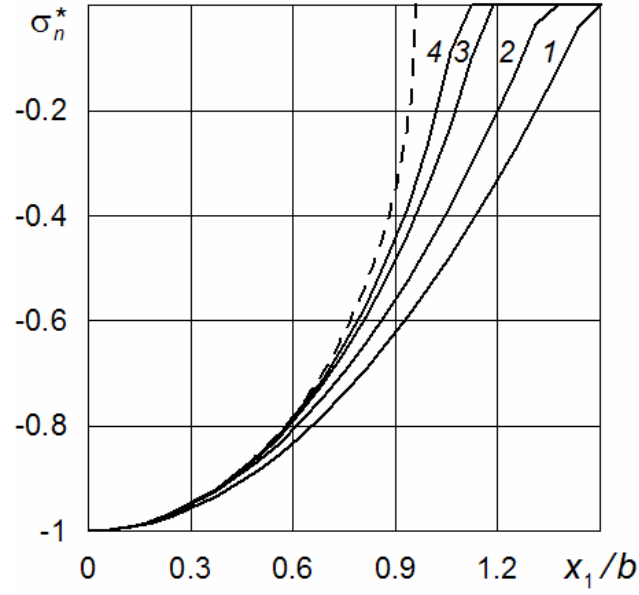


Fig. 3. Dimensionless normal contact stress at different iterations

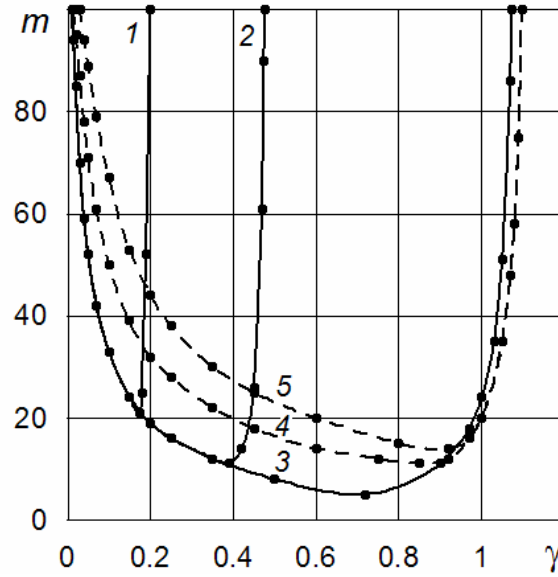


Fig. 4. Total iteration number dependence on iterative parameter γ

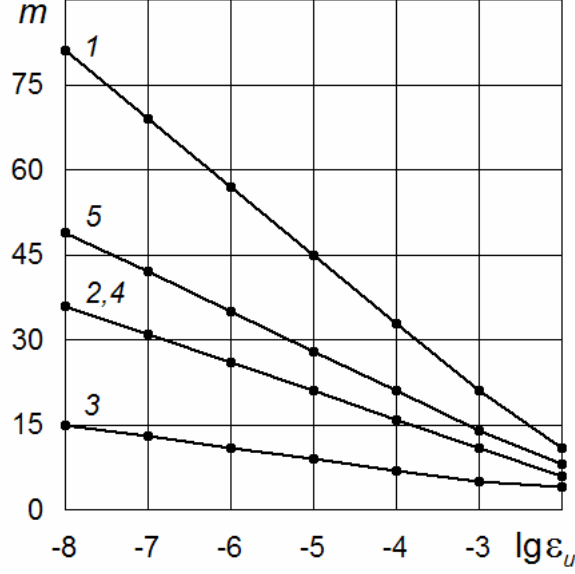


Fig. 5. Total iteration number dependence on logarithmic accuracy

Thus, the convergence rate of stationary Robin–Robin domain decomposition schemes is linear. The parallel Robin–Robin scheme (85) – (86) with the surfaces S_{12}^1 , S_{21}^1 most closed to the real contact area ($S_{12}^1 = S_{21}^1 \approx S_{12}^* \approx \llbracket 0; b \rrbracket$), and the nonstationary parallel Dirichlet–Dirichlet scheme (97) – (98) ($\psi_{12} = \psi_{21} = \chi_{12}^k$), which are represented at curve 3, have the highest convergence rates. These two schemes also have the widest range from which the iterative parameter γ can be chosen. The convergence rate of parallel Neumann–Neumann scheme ($S_{12}^1 = S_{21}^1 = \emptyset$), which is represented at curve 1, is the most slow.

Now let us investigate the convergence of penalty method and its dependence on finite element discretization.

Consider the unilateral contact problem of two isotropic bodies Ω_1 and Ω_2 , one of which has a groove (fig. 6).

The bodies are uniformly loaded by normal stress with intensity q . Each body has length l and height h , and the groove has length b .

The material properties of the bodies are the same: $E_1 = E_2 = E$, $\nu_1 = \nu_2 = \nu = 0.3$. The distance between bodies before the deformation is $d_{12}(\mathbf{x}) = r \{ [1 - (x_1 - l)^2/b^2]^+ \}^{3/2}$, where $r = 0.05b$, $y^+ = \max\{0, y\}$. The possible contact area is $S_{12} = \{ \mathbf{x} = (x_1, x_2)^T : x_1 \in [0; l], x_2 = h \}$.

The problem is solved by nonstationary parallel Dirichlet–Dirichlet domain decomposition scheme (97) – (98) with finite element approximations on triangles.

The penalty parameter is taken as follows

$$\theta = ch \sum_{\alpha=1}^2 (1 - \nu_\alpha)^2 / E_\alpha, \quad (100)$$

where c is dimensionless penalty coefficient. Initial guesses for displacements

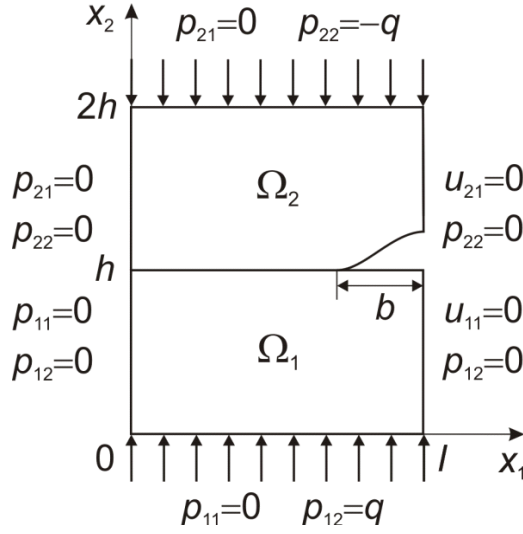


Fig. 6. Unilateral contact of two bodies with a groove

are chosen using the bar model [19]:

$$u_{1n}^0(\mathbf{x}) = \frac{ch}{\theta E_1} \frac{(1 - \nu_1^2)}{(1 + c)} \frac{[d_{12}(\mathbf{x}) - \Delta]^-}{(1 + c)}, \quad \mathbf{x} \in S_{12},$$

$$u_{2n}^0(\mathbf{x}) = \frac{ch}{\theta E_2} \frac{(1 - \nu_2^2)}{(1 + c)} \frac{[d_{12}(\mathbf{x}) - \Delta]^-}{(1 + c)} + \Delta, \quad \mathbf{x} \in S_{21},$$

where $\Delta = q\theta(1 + c)/c$. We use (99) as stopping criterion for iterative process.

For iterative parameter $\gamma \in [0.45; 0.65]$, accuracy $\varepsilon_u = 10^{-3}$, penalty coefficients c and finite element meshes considered below, parallel Dirichlet–Dirichlet scheme, applied to solve this problem, converges in 2–15 iterations.

We investigate the dependence of the quality of numerical solution, obtained by this scheme, on penalty parameter and finite element mesh.

Plots at fig. 7 represent approximations to dimensionless contact stress $\sigma_n^*(\mathbf{x}) = \sigma_{12n}(\mathbf{x})/E$, $\mathbf{x} \in S_{12}$ for bodies with size $h = l = 8b$ and external load $q = 0.01E$, obtained by Dirichlet–Dirichlet scheme for different dimensionless penalty coefficients c at fixed finite element mesh with 64 linear triangular finite elements on each side of possible contact area S_{12} . Curves 1–4 correspond to $c = 0.1, 0.05, 0.01$ and 0.0025 respectively.

Plots at fig. 8 represent approximations to $\sigma_n^*(\mathbf{x})$, $\mathbf{x} \in S_{12}$ for bodies with length $l = 8b$, height $h = 2b$, and external load $q = 0.0075E$, obtained for different dimensionless penalty coefficients c and different finite element meshes.

Curves 1 and 2 at this figure correspond to σ_n^* for dimensionless penalty coefficients $c = 0.1$ and $c = 0.01$ respectively at finite element mesh with 32 linear triangular elements on each side of possible unilateral contact area S_{12} . Curves 3 and 4 correspond to σ_n^* for $c = 0.1$ and $c = 0.01$ respectively, but for finite element mesh with 64 linear triangular elements on each side of S_{12} . Dashed curve at this figure and at fig. 7 represents the exact solution, obtained in [33] for the contact of two half-spaces. Here we see that in spite of the solution, obtained for penalty coefficient $c = 0.1$ at mesh with 32 finite elements

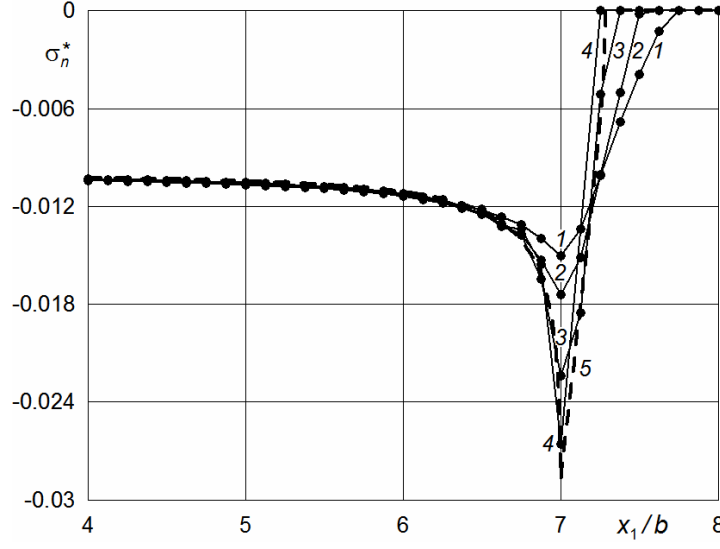


Fig. 7. Normal contact stress σ_n^* for different penalty coefficients (at fixed mesh)

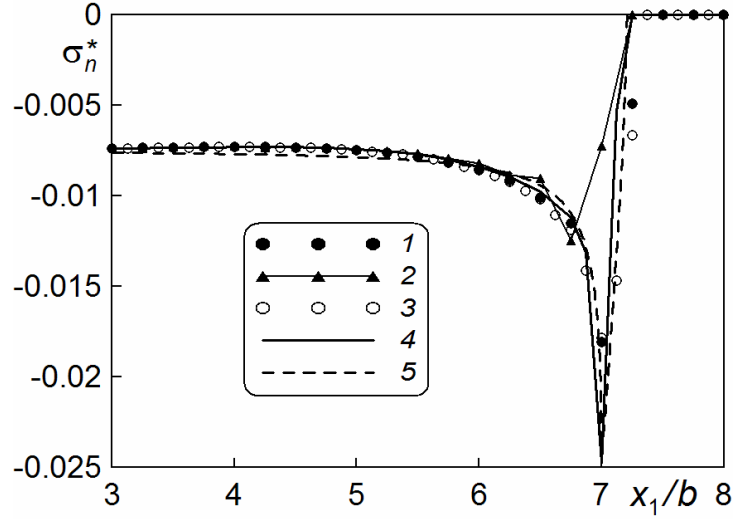


Fig. 8. Normal contact stress σ_n^* for different penalty coefficients and finite element meshes

on each side of S_{12} (curve 1 at fig. 8), the solution obtained for lower penalty coefficient $c = 0.01$ at the same mesh become instable. But if we refine finite element mesh twice for the penalty coefficient $c = 0.01$, then the influence of errors on the perturbation of initial data will decrease, and we shall obtain much better approximation to the exact solution (curve 4 at fig. 8).

Hence, we conclude that for obtaining a nice approximation to the solution, we need to decrease penalty parameter and to refine finite element mesh simultaneously.

8 Conclusions

For the solution of unilateral multibody contact problems of elasticity we have proposed on continuous level a class of parallel Robin–Robin type domain decomposition schemes, which are based on penalty method for variational inequalities and some stationary or nonstationary iterative methods for nonlinear variational equations. In each step of these schemes we have to solve in parallel linear variational equations in subdomains, which correspond to some elasticity problems with Robin boundary conditions on possible contact areas.

We have proved the theorems about the strong convergence and stability of these schemes, and we have shown that the convergence rate of stationary Robin–Robin schemes in some energy norm is linear.

Numerical analysis of proposed domain decomposition schemes has been made for plane two-body contact problems using linear and quadratic finite element approximations on triangles. Convergence rates of different particular domain decomposition schemes have been compared and their dependence on iterative parameter γ has been investigated. The penalty parameter and mesh refinement influence on the numerical solution have been examined. Numerical experiments have confirmed the theoretical results about convergence of these domain decomposition schemes.

The advantages of proposed domain decomposition schemes are their simplicity, and the regularization of the original contact problem because of the use of penalty term. These domain decomposition schemes have only one iteration loop, which deals with domain decomposition and nonlinearity of unilateral contact conditions. In addition, these methods are obtained on continuous level, and they do not depend on any discretization technique.

The generalization of proposed domain decomposition schemes for contact problems of nonlinear elastic bodies are possible.

Moreover, we have built similar domain decomposition schemes for ideal mechanical contact problems (see our work [19]).

The disadvantage of proposed methods is that we have to choose a penalty parameter.

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